Advanced Problem Solving II
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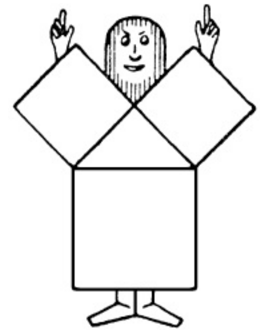
KSU, Fall 2015



Homework 3
Pythagorean Triples and Sums of Squares

Prove all your assertions.

Problem 1. Regular n-gons are constructed on the legs a, b and a hypothenuse c of a right triangle. Check if the sum of the areas of the n-gons on the legs equals the area if the n-gon on the hypothenuse. For this, deduce first a formula for the area of a regular n-gon with side a.

Problem 2. Find all right integer triangles whose area equals half the perimeter. Make sure you showed that there are no other right integer triangles with this property.

Problem 3. Let (a, b, c) be a primitive Pythagorean triple where b is even. Show that b is divisible by 4. Prove this directly without using the theorem that describes all primitive Pythagorean triples.

Problem 4. Let (a, b, c) be a Pythagorean triple. Show that at least one of a and b is divisible by 3. Using this result and the result of the previous problem to prove that the area of an integer right triangle is divisible by 6. Do not use the theorem that describes all primitive Pythagorean triples in this problem.

Problem 5. Let (x, y, z) be a Pythagorean triple. Show that at least one of x or y or z is divisible by 5.

Problem 6.

(a) Show that a positive integer n of the form $n = 4k$ where k is a nonnegative integer can always be written as a difference of squares of two nonnegative integers.

(b) Show that a positive integer n of the form $n = 4k + 1$ where k is a nonnegative integer can always be written as a difference of squares of two nonnegative integers.

(c) Show that a positive integer n of the form $n = 4k + 3$ where k is a nonnegative integer can always be written as a difference of squares of two nonnegative integers.

Problem 7. Show that positive integers n of the form $n = 4k + 2$ where k is a nonnegative integer cannot be written as a difference of squares of two nonnegative integers.

Problem 8. Show that a positive integer n of the form $n = 8k + 7$ where k is a nonnegative integer is not representable as a sum of three squares of nonnegative integers

Problem 9.

(a) Let $a \in \mathbb{Z}$. Prove that

$$a^2 \equiv 0,\ 1,\ \text{or } 4 \bmod 8$$

That is, prove that a square of an integer leaves a remainder of 0, 1, or 4 when divided by 8.

(b) Let m and n be natural numbers. Are there any perfect squares of the form

$$3^m + 3^n + 1?$$

Hint: Look at 3^{2k} and 3^{2k+1} modulo 8. What could $3^m + 3^n + 1$ be equal to modulo 8? In other words, what are the possible remainders after are divided by 8? What remainder can you get when you divide $3^m + 3^n + 1$ by 8? Now use part (a) of the problem.

Project Idea: Read (and understand!) the two squares theorem and the theorem that states when the hypothenuse of a right triangle is a sum of two squares.