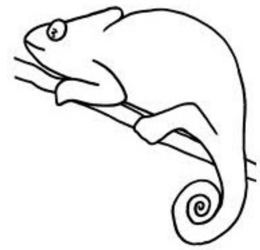
MATH 64091
Jenya Soprunova, KSU



Parity and Chameleons

The only inhabitants of a far away planet are chameleons that come in three colors: green, yellow, and red. Whenever two chameleons of different colors meet they both change to the third color. Let G, Y, and R be the initial numbers of green, yellow, and red chameleons. Our goal is to investigate when a given initial configuration (G, Y, R) is solvable, that is, can be turned into a unicolor configuration.

Exercise 1. Are the configurations (4,5,5), (4,7,5), (4,5,6) solvable?

The first two are easy. Working with the third one we see that something prevents us from turning all the chameleons to one color. Next exercise will help explain why (4,5,6) is not solvable.

Exercise 2. Show that the difference between the numbers of yellow and green chameleons remains the same modulo 3 as chameleons meet and change colors.

Let's check that this is true for one meeting. If we start with (G, Y, R) then depending on which chameleons meet we will end up with one of

$$(G-1, Y-1, R+2), \quad (G-1, Y+2, R-1), \quad (G+2, Y-1, R-1).$$

Then the difference between the numbers of yellow and green chameleons becomes

$$Y-R-3, \; Y-R+3, \text{ or stays } Y-R.$$

That is, the difference changes by a multiple of 3, so it does not change modulo 3. We checked that this is true after one step and it is hence true after finitely many steps.

Note that same statement (and argument) applies to the differences $G-Y$ and $G-R$. Now we are ready to explain why the configuration (4,5,6) is not solvable. If (4,5,6) were solvable, at the end we would have had two of the three numbers equal to zero. Then their difference would also be zero, so initially the difference of the corresponding numbers would have been zero modulo 3, but the initial differences are $4-5=-1$, $4-6=-2$, and $5-6=-1$.

Exercise 3. Show that the configuration (G, Y, R) is solvable if and only if at least two of the given numbers have the same remainder when divided by 3.

Suppose first that (G, Y, R) is solvable. Without loss of generality we can assume that we end up with $(G+Y+R, 0, 0)$. Then by the previous exercise $Y-R$ is the same modulo 3 as $0-0=0$, that is, $Y-R$ is divisible by 3 and hence Y and R have same remainders when divided by 3.

For the other implication, we need to show that whenever two of the three numbers have the same remainder when divided by 3 then the configuration is solvable. Without loss of generality we can assume that those two numbers are Y and R, so $3 \mid (Y-R)$. We can also assume that $Y \geq R$. Then if $Y=0$ then $R=0$ and the configuration is already unicolor, so we can assume that $Y>0$. If $G=0$ we can assume that $R>0$

(otherwise the configuration is already unicolor) and have a yellow chameleon meet a red one:

$$(0, Y, R) \mapsto (2, Y - 1, R - 1).$$

Now it remains to consider the case when G and Y are both positive. Then

$$(G, Y, R) \mapsto (G - 1, Y - 1, R + 2)$$

brings $(Y - R)$ down by 3. After finitely many such steps we will have $Y = R$ and will then have all yellow chameleons and all red chameleons meet and turn green.

Here is an example of how this algorithm works:

$$\begin{aligned}&(2, 16, 1) \mapsto (1, 15, 3) \mapsto (0, 14, 5) \mapsto (2, 13, 4) \mapsto (1, 12, 6)\\ &\mapsto (0, 11, 8) \mapsto (2, 10, 7) \mapsto (1, 9, 9) \mapsto (19, 0, 0).\end{aligned}$$

Homework Problems

Problem 1. Numbers 1 through 50 are written on the board. You can either triple any number or replace any two numbers with their (positive) difference. This can be repeated as many times as you need. Can you end up with a single number which is equal to zero?

Problem 2. Numbers 1 through 50 are written on the board. You can add 1 to any two of them. This can be repeated as many times as you need. Can you make all the numbers equal?

Problem 3. Russ has 101 empty glasses on a tray, 50 of them are upside down. He can pick any four glasses and simultaneously turn them over. Is it possible that after some number of such moves all 101 glasses are turned upside down?

Problem 4. At each vertex of a cube we write either 1 or -1. Then in the middle of each face we write the product of the numbers at the vertices. Can the sum of the obtained 14 numbers be zero? Hint: Find the product of these 14 numbers.

Problem 5. The only inhabitants of a far away planet are chameleons that come in three colors: green, yellow, and red. Whenever two chameleons of different colors meet they both change to the third color. Start with 4 green, 5 yellow, and 6 red chameleons (That is, the configuration is $(4, 5, 6)$.) Can you arrange their meetings so that this configuration changes to

(a) $(4, 2, 9)$?
(b) $(3, 8, 4)$?

Problem 6. The only inhabitants of a far away planet are chameleons that come in four colors: blue, green, yellow, and red. Whenever three chameleons of three different colors meet they all change to the fourth color. Start with 2 green, 2 yellow, 6 red, and 7 blue chameleons. (That is, the configuration is $(2, 2, 6, 7)$.) Can you arrange their meetings so that all the chameleons change to the same color at the end. (That is, the configuration at the end is unicolor.)

Problem 7. Can you turn $(6, 2, 10, 7)$ into a unicolor configuration?

Problem 8. Show that the difference between the numbers of chameleons of two different colors stays the same modulo 4. Use this to show that $(3, 4, 5, 7)$ is not solvable.

Problem 9. Show that if you can turn (G, Y, R, B) into a unicolor configuration, then three of these four numbers are equal modulo 4. Prove all your assertions.

Project idea: Study if the converse of the statement in Problem 9 is true: Whenever at least three out of the four numbers g, y, r, b are equal modulo 4, the configuration (g, y, r, b) can be turned into a unicolor configuration. Clearly, you also need at least three numbers out of the four to be nonzero, if the configuration is not already a unicolor one. Is this enough to require? What about the same problem with a larger number of colors? Another question: When can a configuration (g, y, r) be turned into (G, Y, R)?