MATH 64091
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Games

Example 1. Jack and Jill take turns picking 1, 2, or 3 pebbles from a pile of 10. Whoever takes the last pebble wins. Jack goes first. Who has a winning strategy? Describe the strategy.

Jack starts and wins. He should first pick two pebbles, leaving eight on the table. Whichever amount Jill picks, Jack leaves 4 pebbles on the table after his next move. After Jill's next turn Jack picks up whatever is left and wins.

Here is how one can think about this game in terms of losing and winning positions. We will call the situation with 4 pebbles on the table a losing position since whoever is to go at this point will lose. Similarly, 8 pebbles is a losing position. The remaining positions are winning since one can make a move that will create a losing position for the other player.

Example 2. A rook is placed at the left lower corner of the chessboard. Two players take turns moving the rook any number of squares either to the right or upward. Whoever cannot make a move loses. Who has a winning strategy? Describe the strategy.

The second player wins. The strategy is to always return the rook to the main diagonal.

To come up with this, one can again try to figure out which positions are losing and which ones are winning. The upper right corner $(8, 8)$ is a losing position since whoever is to go now has lost already. Next, all positions of the form (a, b) where either a or b is 8, but not both of them, is a winning position, since one can move the rook to $(8, 8)$ and win. Next, $(7, 7)$ is a losing position since one can only move to a winning position from there. If we keep working on this we will see that all the diagonal positions (a, a) are losing and the rest are winning, so the first player starts out on a losing position and will therefore win. For the second player the strategy is always to move the rook to a loosing position.

Example 3. Tom and Jerry take turns placing quarters on top of a round table so that the quarters do not overlap. Tom goes first. Whoever cannot place a quarter loses. Who has a winning strategy? Describe the strategy.

Tom wins (unless the table is smaller than a quarter). On his first move Tom places a quarter in the exact center of the table. Then wherever Jerry puts his quarter, Tom places his symmetrically with respect to the center of the table.

Example 4. There are a hundred gnomes who have gotten themselves into quite a predicament. They are in the dungeon of a castle of a tyrannical king. Despite the evilness of the king, he has a silver lining in his heart. He has given the gnomes a chance of survival. Here is the offer:

The King tells the gnomes that he will play a game with them. He will line the gnomes up in a single-file row according to their heights with the highest gnome at the back. This way each gnome can see all the gnomes in front of him. Then the king will randomly place either a black or a white hat on each of the gnomes and each of them, starting form the last one in the line will have one chance to guess the color of his hat. If all or all but one of them answer correctly he will let the gnomes go. Otherwise all of them will be executed. Before the game the gnomes can meet with you to discuss the strategy. Can you save the gnomes?

The last gnome should count the number of, say, black hats in front of him and then say "BLACK" if that number is odd and "WHITE" if it's even. With a 50% chance he was able to guess the color of his hat. More, importantly, he passed the vital information to the gnomes in front of him. Now the next gnome can count the number of black hats in front of him and if the parity is the same as the one communicated by the last gnome then his hat is white and if the parity is opposite his hat is black. How does this continue?

Homework Problems

Problem 1. A king is placed at the left lower corner of the 8×8 checkerboard. Two players take turns moving the king one square to the right, or upward, or diagonally right and upward. Whoever cannot make a move loses. Determine who has a winning strategy and describe the strategy.

Problem 2. Heather and Scott have a rectangular 4×8 chocolate bar with lines for breaking it into 32 small square pieces. They noticed that the bottom right square of chocolate is spoiled and cannot be eaten. Heather breaks the bar along one of the lines and hands the piece containing the spoiled square to Scott. Then Scott breaks the piece handed to him along one of the lines and gives back the piece containing the spoiled square to Heather. They do this until one of them gets the lone spoiled square and loses. Who has a winning strategy? Describe the strategy.

Problem 3. Ann and Mariana play the following game. There is a pile of 20 coins on the table. They take turns taking some number of coins from the pile. The rules are: (1) one can take at most 5 coins on the first move; (2) one has to take at least one coin on every move; (3) no one can take more coins than their opponent took on his/her previous move. Whoever takes the last coin wins. Ann goes first. Describe a winning strategy for Ann.

Problem 4. Show that if in the game of Tic-Tac-Toe the first player plays in the corner and then the second player does not play in the center, then the first player has a winning strategy. Describe the strategy.

Problem 5. Starting with the following configuration of tiles

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two players take turns picking any nonzero number of tiles from a single row. Whoever picks the last rock is the winner. Which player has a winning strategy? (Make sure you do not use any theoretical results about a general version of this game.)

Problem 6. A box contains 300 matches. Two players take turns removing no more than half the matches from the box. Whoever cannot make a move loses. Which player has a winning strategy? Describe the strategy.

Problem 7. There are two piles of galleons on the floor, one of 9 and the other of 10 galleons. Harry and Ron take turns picking either any number of galleons from one pile or two galleons, one from each pile. Whoever takes the last galleon wins. Harry goes first. Who has a winning strategy? Describe the strategy.