MATH 64091
Jenya Soprunova, KSU

Group Testing

In all these problems you can use a Geiger counter that can detect if there is a radioactive potato in a pile of potatoes, but cannot tell how many there are.

Example 1. Provide an algorithm for finding one radioactive potato in a pile of four in two checks.

Check two potatoes. If the result is positive then the radioactive one is among these two. If it is negative, then the bad potato is among the unchecked two potatoes. Either way, we have detected two potatoes one of which is radioactive. To determine which one, we just need to check one of them.

Example 2. Provide an algorithm for finding two radioactive potatoes in a pile of ten in six checks.

Break the potatoes in three groups, two of four potatoes each and one with two potatoes. Use two checks to check two piles of four. If both results are positive, then each of these piles has one bad potato and we need two more checks for each of the piles to find the bad potatoes using previous problem.

If one result is positive and the other is negative, check the third pile with two potatoes. If the result is positive, we have one positive pile of 4 and one positive pile of 2, so each of them has one bad potato. We need two more checks for the pile of 4 (previous problem) and one more check for the pile of 2, so we are done in 6 checks.

If the third pile is negative, then the positive pile of 4 has two bad potatoes, which we can find checking three individual individual potatoes in this pile and therefore finishing the task in six checks total.

We next address the question of how to tell if one can find k radioactive potatoes in a pile of N potatoes in n checks. We will show that if the numbers of configurations $\binom{N}{k} > 2^n$ then there is no algorithm for determining the radioactive potatoes. We do this by induction on n. For the base step we need to show that if $\binom{N}{k} > 2$ then we cannot find the bad potatoes in one check. However many potatoes we check we have two possible answers, positive and negative, while the number of possible configurations is larger than 2, so at least two configurations will correspond to the same answer and it is impossible to tell them apart. Now we work with a general $n \in \mathbb{N}$. After first check, we can break all configurations in two groups, the ones that correspond to the positive result and the ones that correspond to the negative result. Since the overall number of configurations is greater than 2^n, the larger of these two groups will be of size greater than 2^{n-1} and by induction assumption we will not be able to finish the task in 2^{n-1} checks.

Example 3. Explain why there is no algorithm for finding two bad potatoes in a pile of 10 in five checks.

If we use five checks, then the number of possible configurations of bad potatoes is $\binom{10}{2} = 45 > 2^5 = 32$, so we would not be able to distinguish all possible configurations.

Homework Problems

Problem 1. Provide an algorithm for finding two radioactive potatoes in a pile of 12 in 7 checks.

Problem 2. Provide an algorithm for finding one radioactive potato in a pile of 2^n potatoes using n checks. Explain why $n-1$ checks would not suffice.

Problem 3. Given two piles of five potatoes, with one radioactive potato in each, provide an algorithm for finding the radioactive potatoes in 5 checks. Explain why 4 checks would not be enough. Hint: On your first check, test two potatoes, one from each pile.

Problem 4. Let $m, n \in \mathbb{N}$. Given two piles, one with 2^m+1 potatoes and one with 2^n+1, with one radioactive potato in each, provide an algorithm for finding the radioactive potatoes in $m+n+1$ checks, except for the case when $m=n=1$. For $m=n=1$, explain how many checks are needed and provide an algorithm for finding the radioactive potatoes.

Problem 5. Given two piles, one with 5 potatoes and one with 6, with one radioactive potato in each, provide an algorithm for finding the radioactive potatoes in 5 checks. Hint: On your first check, test one potato from the first pile and two from the second.

Problem 6. Given two piles, one with 7 potatoes and one with 9, with one radioactive potato in each, provide an algorithm for finding the radioactive potatoes in 6 checks.

Problem 7. Show that there is no algorithm for finding 2 bad potatoes in a pile of 6 in 4 checks. Note that if you compare the number of configurations $\binom{6}{2}$ with 2^4, as we did in Example 3, there will be no contradiction. Instead, consider all possible first checks and depending on whether the result is positive or negative, compare the number of possible configurations with 2^3.

Problem 8. Provide an algorithm for finding two radioactive potatoes in a pile of 15 potatoes using 7 checks. Hint: Start with checking a pile of five potatoes. If the result is positive, check one of these five potatoes (call it A) together with four out of the remaining 10. If the result is positive again, check potato A.