MATH 64091
Kent State University

Proving Inequalities

Example 1. Show
$$\frac{x^2+y^2}{2} \geq xy$$
for every value of x and y.

Here is a chain of equivalent inequalities.
$$\begin{aligned} \frac{x^2+y^2}{2} &\geq xy \\ x^2+y^2 &\geq 2xy \\ x^2-2xy+y^2 &\geq 0 \\ (x-y)^2 &\geq 0 \end{aligned}$$

The last inequality in this chain obviously holds for all values of x and y. Note that the inequality turns into equality if and only if $x=y$.

Example 2. Show that the geometric mean of two positive non-negative a and b does not exceed their arithmetic mean:
$$\frac{a+b}{2} \geq \sqrt{ab}.$$
When does this inequality turn into equality?

Here is a chain of equivalent inequalities reducing the original inequality to a trivial one.
$$\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab} \\ a+b &\geq 2\sqrt{ab} \\ (\sqrt{a})^2-2\sqrt{a}\sqrt{b}+(\sqrt{b})^2 &\geq 0 \\ (\sqrt{a}-\sqrt{b})^2 &\geq 0 \end{aligned}$$

This turns into equality when $\sqrt{a}=\sqrt{b}$, that is, $a=b$.

Example 3. Let a, b be non-negative numbers. Show that $a \geq b$ if and only if $a^2 \geq b^2$.

Here we have $a^2-b^2=(a-b)(a+b)$ and $a+b \geq 0$. To prove the direct implication we suppose that $a \geq b$. Then $a-b \geq 0$ and hence $a^2-b^2=(a-b)(a+b) \geq 0$. For the reverse implication, suppose that $a^2 \geq b^2$. Then $a^2-b^2 \geq 0$ and hence $a-b=\frac{(a^2-b^2)}{(a+b)} \geq 0$, which implies that $a \geq b$. Note that we can only divide by $a+b$ if $a+b$ is non-zero, but if $a+b=0$ both a and b would have to be zero (by assumption a and b are non-negative) and in this case we trivially have the desired conclusion that $a \geq b$.

Example 4. Show that the arithmetic mean of two numbers a and b does not exceed their *quadratic mean*:
$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}.$$

To get rid of the radical it would be nice to square both sides, but we have to be careful here. For example, squaring a true number inequality $-3 < 2$ we obtain $9 < 4$! This wouldn't happen if both sides of the original inequality were non-negative as we showed in previous example.

Let's look at the inequality for the means. The right-hand side is always greater than or equal to zero, while the left-hand side could be negative, but if it is negative, the inequality is obvious – we have something negative on the left which is obviously less than something non-negative on the right. Now, let's assume that both sides are non-negative and square the inequality:

$$\begin{aligned} \frac{a+b}{2} &\leq \sqrt{\frac{a^2+b^2}{2}} \\ \frac{a^2+2ab+b^2}{4} &\leq \frac{a^2+b^2}{2} \\ a^2+2ab+b^2 &\leq 2a^2+2b^2 \\ a^2-2ab+b^2 &\geq 0 \\ (a-b)^2 &\geq 0 \end{aligned}$$

Notice that combining two last examples, for positive a and b we get an inequality for all three means:

$$\sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

which turns into equality if and only if $a = b$.

Example 5. Show that for all positive x, y we have

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}.$$

Multiplying both sides by $xy(x+y)$ we get:

$$\begin{aligned} y(x+y) + x(x+y) &\geq 4xy \\ x^2+y^2+2xy &\geq 4xy \\ (x-y)^2 \geq 0 & \end{aligned}$$

here last inequality clearly holds true for all values of x and y.

Example 6. Show that for all x, y, z we have

$$x^2+y^2+z^2 \geq xy+yz+zx.$$

When does this inequality turn into equality?

Multiplying both sides by 2 we get:

$$\begin{aligned} x^2+y^2+y^2+z^2+x^2+z^2 &\geq 2xy+2yz+2xz \\ (x-y)^2+(y-z)^2+(z-x)^2 &\geq 0 \end{aligned}$$

This sum equals zero if and only if $x - y = 0,\ y - z = 0,\ z - x = 0$, that is, $x = y = z = 0$.

Homework Problems

Problem 1. Prove for positive a, b, c, d

$$\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}.$$

Problem 2. Show for all real x and y that

$$x^2 + y^2 + 1 \geq xy + x + y.$$

Problem 3. Given that $(a+b+c) = 0$ show that $ab + bc + ac \leq 0$.

Problem 4. Show that for all $a, b, c \geq 0$ we have

$$(a+b)(a+c)(b+c) \geq 8abc.$$

Problem 5. Show that for any real numbers x_1, x_2, y_1, y_2

$$(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2).$$

Problem 6. Show that for $a, b, c \geq 0$

$$ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}.$$

Hint: Multiply this through by 2 and then collect the terms with a, b, and c.

Problem 7. Show that for any choice of real numbers a, b, c

$$a^4 + b^4 + c^4 \geq abc(a+b+c).$$

Hint: We proved in class that for any real numbers x, y, z

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

Use this inequality twice.

Problem 8. Show that for positive a, b, c

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} \geq a + b + c.$$