MATH 64091
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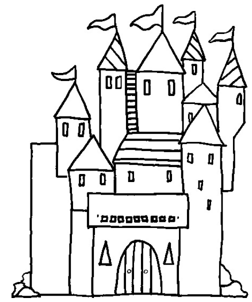

Monovariants

The key to solving problems in this set is to consider some quantity M (called a monovariant) that changes in a predictable way.

Example 1. Two players take turns breaking up a 3×4 chocolate bar along the division lines. Whoever breaks off the last piece wins and gets to eat the chocolate. Does any of the players have a winning strategy?

Let M be the number of pieces of chocolate. At the beginning of the game $M=1$. At each turn M goes up by 1. At the end of the game $M=12$. Hence, whichever way the game is played, the players will take 11 turns, so first player will win.

Example 2. A thousand people reside in the rooms of a 100-room mansion. Each minute, as long as not everyone is in the same room, somebody walks from a room to a different room with at least as many people in it. Prove that eventually all the people will gather in one room. (One can walk to any other room, not necessarily a neighboring one, as long as the number of people there is the same or larger than the number of people in their current room.)

Let $a_1, a_2, \dots, a_{100}$ be the numbers of people in each of the rooms. We need to put all of these numbers together to produce one number. The most obvious choice is to consider the sum $M = a_1 + a_2 + \cdots + a_{100} = 1000$, which is not very helpful, since M will stay constant as people walk from room to from and will not help as solve the problem. Instead, let M be the sum of squares of the a_i, that is,

$$M = a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2.$$

Let's see how this number changes as a person walks from one room to another. Without loss of generality, we can assume that $a_1 \le a_2$ and a person walks from room 1 to room 2. Then the new value of M would be

$$M_{\text{new}} = (a_1-1)^2 + (a_2+1)^2 + a_3^2 \cdots + a_n^2.$$

Let's show that $M_{\text{new}} > M$. For this we need to check that

$$(a_1-1)^2 + (a_2+1)^2 > a_1^2 + a_2^2.$$

This inequality is equivalent to

$$a_1^2 - 2a_1 + 1 + a_2^2 + 2a_2 + 1 > a_1^2 + a_2^2,$$

which is in turn equivalent to $a_1 < a_2 + 1$ or $a_1 \le a_2$, which is exactly our assumption. We have checked that at each step M goes up. Since M is bounded from above (by, for example, a very rough bound $100 \cdot 1000^2$ where we used the fact that each $a_i \le 1000$), the process will stop, which can only happen when all the people are in the same room.

Homework Problems

Problem 1. Three players take turns breaking up a 3D $m \times n \times l$ chocolate bar along the division lines. Whoever breaks off the last piece wins and gets to eat the chocolate. Does any of the players have a winning strategy?

Problem 2. Provide a different solution to Exercise 2 that makes use of

$$M = \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_{100} + 1}.$$

Explain why we cannot define M to be $\sum \frac{1}{a_i}$.

Problem 3. On a 10 by 10 square field nine 1 by 1 squares are contaminated with poison ivy. Whenever a 1 by 1 square has at least two contaminated neighbors it also becomes contaminated. Prove that poison ivy cannot spread to the whole field. (Two squares are neighbors if they share either a horizontal or a vertical segment.) Hint: Let M be the perimeter of the contaminated region.

Problem 4. All pairwise distances between cities in a country are distinct. A tourist leaves city A and travels to city B, which is the farthest away from A. From B he travels to city C, which is the farthest away from B, and so on. Given that C is different from A, show that the tourist will never come back to A. Hint: Let M be the length of the last segment that the tourist has traveled.

Problem 5. Each of n countries is either a democracy or a monarchy. Each year, one of the countries that has more neighbors with a different than the same form of government changes its form of government. For example, if a democratic country A has three democratic neighbors and five monarchic ones, it can change its system to monarchy. (a) Show that such changes cannot go on forever. (b) Draw a connected map where no changes can be made but not all countries have the same system. Hint for (a): Let M be the number of pairs of neighboring countries that have the same form of government.

Problem 6. A group children stands in a circle. Each of them has an even number of candies. At each step, each of the children passes half of their candy to a child on their right. If after this, some of the children end up with an odd amount of candies, an adult who supervises the game gives each of them an extra candy from his supply. Show that after a while all the children will have the same number of candy. Hint: let M_1 be the largest number of candy one of the children has and M_2 be the smallest number. How do M_1 and M_2 change during the game?