MATH 64091
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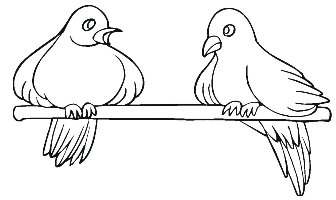

The Pigeonhole Principle

The pigeonhole principle sounds very simple and obvious: If we put $N+1$ pigeons into N pigeon holes, there will be at least one hole with at least 2 pigeons. Similarly, if we must put $Nk+1$ pigeons into N pigeon holes, at least one of the will contain at least $N+1$ pigeons. Indeed, otherwise the overall number of pigeons would be at most Nk. This simple observation helps solve a range of very different problems.

Example 1. Over one million people live in the Cleveland area. Show that there are two people with exactly the same numbers of hairs on their heads. (It is known that nobody can have more than a million hairs on their head.)

We use the pigeonhole principle here. The pigeonholes are the possible numbers of hairs, so we have 1000001 pigeonholes and over 1000001 pigeons (people) to put in those pigeonholes. Since the number of pigeons is larger than the number of holes, there will be at least two pigeons in at least one of the holes. That is, there will be at least two people with the same numbers of hairs on their heads.

Example 2. You have black and white socks in your drawer. The room is pitch dark. How many socks do you need to pull from the drawer so that there are two of the same color? (You don't care if the pair is white or black.) What if you have black, white, and red socks in your drawer? What if you have black, white, red, and green socks in your drawer and you want to get five socks of the same color?

The answer to the first question is three. Out of three socks at least two will be of the same color. If we picked only two socks they could be of opposite colors. If we have black, white, and red socks in the drawer, we would need to pick four socks. Finally, the answer to the last question is 17. We have four pigeonholes (colors of socks) and 17 pigeons, so at least five pigeons would have to end up in the same pigeonhole. That is, there would be five socks of the same color. If we pick 16 socks, it could be the case that we picked four socks of each of the four colors.

Example 3. What is the largest number of squares on an 8×8 chessboard which can be colored green, so that in any three squares that form an L-tromino at least one square is not colored green. (The L-tromino may be rotated through a multiple of 90 degrees.)

If we color every other row on the board green, there would be 32 green squares and in any three squares that form an L-tromino at least one square is not colored green.

Let's show that 32 is the largest number of squares which can be colored green, so that the condition is satisfied. Assume that one can actually place at least 33 green squares in the required fashion.

Break the board into sixteen 2 by 2 squares. These 16 two by two squares are our holes. We have more than 33 pigeons (green squares). Hence there will be a two by two square with at least 3 green squares, which would have to form a green L-tromino.

Hence there are at most $16\cdot 2 = 32$ green squares.

Example 4. Given 14 integers, show that two of them can be chosen so that their difference is divisible by 13.

The pigeonholes now are the possible remainders after division by 13. There are 13 such holes, 0 through 12. Hence at least two of the fourteen integers will end up in the same pigeonhole, that is, will give same remainder after they are divided by 13, so their difference would be divisible by 13.

Homework Problems

Problem 1. A four by four table is filed with sixteen numbers, each of which equals -1, 0, or 1. Show that among the ten possible sums along the rows, columns, and diagonals at least two are the same.

Problem 2. Given seven integers show that one can pick three of them so that their sum is divisible by 3.

Problem 3. Given 8 different integers, all between 1 and 15 (1 and 15 are allowed), show that at least three pairs of them have the same (positive) difference. (The pairs need not be disjoint as sets. For example, (1,4), (4, 7), and (5,8) would work.)

Problem 4. Show that there exists an integer all of whose digits are ones such that it is divisible by 2017.

Problem 5. Show that in any group of 10 people, there are at least two who have an identical number of friends within the group. (If person A is friends with person B, person B is also friends with person A.)

Problem 6. What is the smallest number of squares on an 8×8 checkerboard which can be colored green, so that among any three squares that form an L-tromino at least one square is green?

Problem 7. Each vertex of a cube is marked with either 0 or 1. Each face is then marked with the sum of the integers written at its vertices. Is it possible that all the numbers on the faces are distinct? What if each of the vertices is marked with either 1 or -1?

Problem 8. Prove that there exist two different powers of 7 that differ by a multiple of 2017.

Problem 9. Color all the points with integer coordinates in the plane green. Show that whichever way you pick five of these green points, there will be two among them such that the segment connecting these two points has a green midpoint.

Bonus 1. Let $a_1, \ldots, a_{2017}$ be positive integers. Show that you can always pick a few of them so that their sum is divisible by 2017.