MATH 64091
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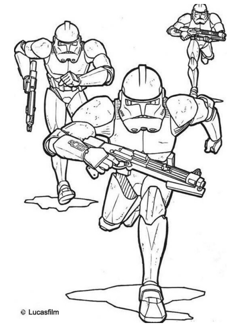

The Pointless Machine and Escape of the Clones

The Pointless Machine that operates on ordered pairs of positive integers (a, b) has three modes: In *Mode 1* the machine adds 1 to both coordinates, e.g, $(3, 17)$ produces $(4, 18)$. In *Mode 2*, the machine takes pairs of even numbers and divides both coordinates by 2, e.g, $(6, 14)$ produces $(3, 7)$. In *Advanced Mode* the machine takes two pairs of numbers (a, b) and (b, c) and glues them together to produce (a, c), e.g., $(3, 5)$ and $(5, 7)$ produce $(3, 7)$. The machine remembers and can use any of the pairs that have been previously produced or input into the machine.

Exercise 1. Initially, (3,17) is available to the Pointless Machine. Can you produce the following pairs using the machine?

(a) (86,100)
This is easy, we just need to apply Mode 1 eighty three times.

(b) (20,27)
For this, we apply Mode 1 to get(4,18), then apply Mode 2 to get (2,9) and then finally apply Mode 1 eighteen times.

(c) (1,8)
We already know how to get (2,9). Applying Mode 1 seven times we will get (9,16). Now we use the Advanced Mode and obtain (2,16) from (2,9) and (9,16). Finally, applying Mode 2 to (2,16) we obtain (1,8).

(d) (7,13)
To show that this one is impossible to obtain we observe that in the initial pair we have $17 - 3 = 14$, which is divisible by 7. Now, if the machine can only use pairs with the difference divisible by 7, it will only produce pairs with the same property. We need to check that this is true for each of the three modes.

In Mode 1 we have $(a, b) \mapsto (a + 1, b + 1)$. If $7 \mid b - a$ then we also have $7 \mid ((b + 1) - (a + 1))$, since this difference equals $(a - b)$. In Mode 2, $(2a, 2b) \mapsto (a, b)$ and $7 \mid (2a - 2b) = 2(a - b)$ implies that $7 \mid (a - b)$. Finally, in the Advanced Mode (a, b) and (b, c) produce (a, c). If we have that $7 \mid b - a$ and $7 \mid (c - b)$, we also have $7 \mid (c - a)$ since $c - a = (c - b) - (b - a)$.

Exercise 2. Prove that the Pointless Machine always preserves any odd divisor of the difference of the starting pair and that if $0 < a < b$ in the starting pair (a, b) then same is true about all the pairs produced.

The argument here is exactly the same as in the exercise above, we only need to replace 7 with an odd divisor d. To explain the second statement we need to check again that all three modes preserve the inequality.

Exercise 3. Describe all the pairs that can be produced from the pair $(3, 17)$.

Let's show that we can produce all the pairs of positive integers (a, b) such that $a < b$ and $7 \mid (b - a)$. From the previous problem we know that all the produced pairs would have to satisfy these properties. It remains to show that all pairs (a, b) such that $a < b$ and $7 \mid (b - a)$ can be produced.

Since $7 \mid (b - a)$ and $a < b$ we have $b - a = 7k$ for some positive integer k, that is, $(a, b) = (a, a + 7k)$.

We want to provide an explicit algorithm of how to obtain a pair $(a, a + 7k)$ where k and a are positive integers. From Exercise 1 we know how to produce $(1, 8)$. Next applying Mode 1 multiple times we can produce $(8, 15), (15, 22), \ldots, (7k-6, 7k+1)$. Now using Advanced mode we can glue these pairs together to produce $(1, 7k + 1)$. Finally, applying Mode 1 $(a - 1)$ times we produce $(a, a + 7k)$.

In the Escape of Clones game, the first quadrant is divided into unit squares by horizontal and vertical lines at the positive integers. Three clones are placed in the shape of an L-tromino in the bottom left-most squares. At each step you can erase a clone and replace it with two copies in two adjacent squares, one directly above and the other directly to the right, as long as those squares are currently unoccupied. Let's recall how we showed in class that it is impossible to free all the clones from the L-tromino prison.

We label each square with a pair of integers starting with $(0, 0)$. That is, the L-tromino prison consists of the three squares $(0, 0)$, $(1, 0)$, and $(0, 1)$. We would like to assign a number to each of the squares so that the sum of numbers assigned to the squares occupied by clones does not change as the clones move. Let's start with assigning 1 to the square $(0, 0)$. A clone at $(0, 0)$ may split in two and move to $(1, 0)$ and $(0, 1)$, so it makes sense to assign $1/2$ to each of those squares. Next, we want to assign $1/4$ to the squares $(2, 0), (1, 1), (0, 2)$. Continuing in this fashion, we assign $1/2^{a+b}$ to every square (a, b).

Exercise 4. Assign $1/2^{a+b}$ to every square (a, b) in the first quadrant. Show that when a move is applied to a clone anywhere, the sum of numbers assigned to occupied squares before and after the move are equal.

Let us assume that there is a clone in the square (a, b). Then after a move we have two clones at the positions $(a+1, b)$ and $(a, b+1)$. The sum of numbers before the move is $1/2^{a+b}$, and after the move it is $1/2^{a+b+1} + 1/2^{a+b+1}$, which is the same as $1/2^{a+b}$.

Exercise 5. Show that all the clones clones cannot escape from an L-tromino prison. That is, we want to show that if in the beginning in the game we have 3 clones positioned at the squares $(0, 0), (1, 0)$, and $(0, 1)$, then after a sequence of moves we cannot end up with a configuration where no clones occupy any of those three positions.

The sum of numbers assigned to the squares in the beginning of the game is

$$1 + 1/2 + 1/2 = 2.$$

Hence we should have the same sum at the end of the game. Let's compute the sum of all the numbers assigned to the squares in the quadrant. We will do this row by row.

The sum in the first row is

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots = \frac{1}{1 - 1/2} = 2.$$

The sum in the second row is one half of the sum in the first row, the sum in third one is one have of the second, and so on. Hence the overall sum is

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 4.$$

and the sum of all the numbers outside of the L-tromino jail is 4-2=2. If it were possible to free all the clones, they would end up on finitely many squares outside the jail, so the sum of the occupied squares after the game would be strictly less than 2, which would not match the initial sum of 2 and therefore it's impossible to free the clones.

Homework Problems

Problem 1. Initially $(3,18)$ was available to the Pointless Machine. Describe all the pairs that can be produced by the machine. In your write-up follow the plan: 1) Give a description of all the pairs that can be produced. 2) Show that any pair produced by the machine has to satisfy your description. 3) Show that any pair that satisfies your description can be produced by the machine. Make sure that your solutions has these three parts. Prove all your assertions.

The Even More Pointless Machine operates on ordered pairs of positive integers (a,b) and has three modes: In *Mode 1* the machine adds 1 to both coordinates, e.g, $(3,17)$ produces $(4,18)$. In *Mode 2*, the machine takes pairs of numbers that are divisible by 3 and divides both coordinates by 3, e.g, $(6,18)$ produces $(2,6)$. In *Advanced Mode* the machine takes two pairs of numbers (a,b) and (b,c) and glues them together to produce (a,c), e.g., $(3,5)$ and $(5,7)$ produce $(3,7)$. The machine remembers and can use any of the pairs that have been previously produced or input into the machine.

Problem 2. If initially $(4,19)$ was available to the The Even More Pointless Machine, can it produce

(a) $(22,27)$
(b) $(6,26)$
(c) $(3,17)$
(d) $(2,37)$?

Problem 3. Describe all the pairs the The Even More Pointless Machine can produce if the starting configuration is $(4,19)$. In your write-up follow the plan: 1) Give a description of all the pairs that can be produced. 2) Show that any pair produced by the machine has to satisfy your description. 3) Show that any pair that satisfies your description can be produced by the machine. Make sure that your solutions has these three parts. Prove all your assertions.

The Simple Pointless Machine operates on positive integers (NOT pairs of integers, just integers) and has only two modes. In *Mode 1* the machine adds 3 to the number, e.g., 3 produces 6. In *Mode 2*, the machine takes even numbers and divides them by 2, e.g., 8 produces 4. The machine remembers and can use any of the numbers that have been previously produced or input into the machine.

Problem 4. If initially 17 was available to the Simple Pointless Machine, describe all the numbers that it can produce. In your write-up follow the plan: 1) Give a description of all the integers that can be produced. 2) Show that any integer produced by the machine has to satisfy your description. 3) Show that any integer that satisfies your description can be produced by the machine. Make sure that your solutions has these three parts. Prove all your assertions.

Problem 5. If initially a positive integer a was available to the Simple Pointless Machine, describe all the numbers that it can produce. In your write-up follow the plan: 1) Give a description of all the integers that can be produced. 2) Show that any integer produced by the machine has to satisfy your description. 3) Show that any integer that satisfies your description can be produced by the machine. Make sure that your solutions has these three parts. Prove all your assertions.

Problem 6. Show that for any $r \neq 1$ we have

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Show that if $|r| < 1$ then for the sum of the infinite series we have

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}.$$

Memorize both formulas.

Problem 7. A single clone is placed in the bottom left-most square. The prison encloses the following 10 squares: four leftmost squares in row one, three leftmost squares in row two, two leftmost squares in row three, and, finally, one leftmost square in row 4. Show that there will always be at least one clone in the prison. Hint: Your solution will be very similar to the one we did in class.

Problem 8. A single clone is placed in the bottom left-most square. The prison encloses the following 6 squares: three leftmost squares in row one, two leftmost squares in row two, and one leftmost squares in row three. Show that there will always be at least one clone in the prison. Hint: This is similar to the previous problem , but you will have to make an observation that there is at most one clone in row 1 and at most one clone in column 1 (why?).

Project Idea: Read a paper by Chung, Graham, Morrison, and Odlyzko in 1995 Monthly at www.math.ucsd.edu/∼ronspubs/95_03_pebbling.pdf. Starting with one clone in the leftmost bottom square the paper describes all the inescapable prison shapes.