MATH 64091
KSU, Fall 2016
Jenya Soprunova

Fake Coins and a Balance Scale
Examples and Homework

In the solutions please avoid using phrases like "in the worst-case scenario". Often, it's hard to tell which case will turn out to be the "worst", so do not neglect even the simplest or most unlikely cases. Write your solutions in complete sentences. Try to organize your writing so that your solution is easy to read and understand. Explain all the details.

Example 1. One of the three identically looking coins is fake. It is known that the fake coin is heavier than the other two. How can you decide which one of the three is fake using a balance scale? How many weighings would you need?

Solution. One weighing. Place one coin on one side of the balance scale, one on the other, and put one coin aside. If the scale balances, the coin that we put aside is fake. If one of the sides of the scale is heavier, the fake coin is on that side. □

Example 2. One of the nine identically looking coins is fake. It is known that the fake coin is heavier than the other eight. How can you determine, in two weighings on a balance scale, which coin is fake?

Solution. First weighing: three coins aside, three on each side of the scale. This way you will determine three coins which have a fake coin among them. Second weighing: Problem 1. □

Example 3. One of the 27 identically looking coins is fake. It is known that the fake coin is heavier than the other 26. How can you determine, in three weighings on a balance scale, which coin is fake?

Solution. First weighing: 9 coins aside, 9 on each side of the scale. This way you will determine 9 coins which have a fake coin among them. Now the problem is reduced to Example 2. □

Example 4. There is a possibility that one of the ten identically looking coins is fake. The fake coin differs in weight from the original ones. How can you decide using a balance scale if there is indeed a fake coin among these 10 coins? How many weighings would you need to determine that?

Solution. One weighing. Put 5 coins on one side of the scale, 5 on the other. If the scale balances, there are no fake coins. If one of the sides is heavier than the other, one of the 10 coins is fake. □

Example 5. One of the 99 identically looking coins is fake. The fake coin differs in weight from the original ones, but it is not known whether the fake coin is lighter or heavier than the rest. How can you determine, in two weighings, if the fake coin is lighter or heavier? What if you had 101 coins?

Solution. Break the coins into 3 piles of 33. Compare two piles against each other. If their weight match, it means the fake coin is among the 33 coins put aside and by comparing that remaining pile against one of the other two we will be able to see if the fake coin is lighter or heavier.

If the weights do not match (that is, the fake coin is on the scales), compare the lighter pile against the pile put aside. If the weights match, it means the fake coin is heavier that the original ones, if the weights don't much the fake coin is lighter.

Next, we deal with 101 coins. Break these coins in two piles of 50 and put one coin aside. Weigh these two piles against each other. If the weights match, the remaining coin is fake and we can figure out if it's heavier or lighter to comparing to any coin from the two piles.

If the weights do not match, pick, say, the lighter pile, break it into two piles of 25 and compare the two piles against each other. If the weights match, it means there is no fake coin among these 50, so the fake one is heavier than the original ones. If the weights do not match, it means there is a fake coin among these 50, so the fake one is lighter than the original ones. □

Example 6. Given a 3-gallon and a 5-gallon containers, a funnel, and a water tap, pour exactly 4 gallons of water into the 5-gallon container. Would you be able to do this if the containers were of volume 9 and 12 gallons?

Solution. Fill up the 3-gallon container, pour these 3 gallons in to the 5-gallon container. Fill up the 3-gallon container again and pour the water from this container to the 5-gallon one until the 5-gallon container is full. You will have 1 gallon left in the 3-gallon container. Empty the 5-gallon container and then pour 1 gallon from the 3-gallon container to the 5-gallon one. Finally, fill up the 3-gallon container and pour these 3 gallons to the 5-gallon container, which will now contain 4 gallons of water.

The answer to the second question is no since whichever operation we do with container whose volume i liters is a multiple of 3 the measured volume will also be a multiple of 3, so we would never be able to measure 4 gallons of water. □

Homework Problems

Problem 1. Same as Example 2, but it is not known if the fake coin is heavier or lighter than the rest and you are allowed 3 weighings.

Problem 2. One out of four identically looking coins is fake. Find the fake coin using a balance scale twice.

Problem 3. You have 13 apples and a scale that allows you to measure the weight of any two apples combined together. Find a way to compute the overall weight of all the apples using the scale 8 times.

Problem 4. You have a 12 gallon bucket full of milk and two empty buckets measuring 5 and 7 gallons. Use these three buckets two split the milk in two halves (between the 12 and 7 gallon buckets).

Problem 5. Find a set of three weights which you can use to measure any whole number of pounds of sugar from 1 to 10 on a balance scale. Describe how you would measure those amounts of sugar using the weight that you suggest. (You can put the weights on both sides of the scale.)

Problem 6. Two of the six coins are counterfeit and are lighter than the genuine ones. How can you determine, in three weighings on a balance scale, which coins are counterfeit ?

Problem 7. Two of the five coins are counterfeit. One of those two is lighter and the other is heavier than the genuine ones. How can you determine, in three weighings on a balance scale, which coins are counterfeit?

Problem 8. There are 64 stones of different weights (no two stones weigh the same). Find, in 94 weighings on a balance scale, the heaviest and the lightest of the stones.

Problem 9. You have 10 bags of coins, 10 coins per bag, 10 grams per coin, but in one of the ten bags all the coins are fake each weighing only 9 grams. Use a **digital** scale only once to find out which bag contains fake coins.

Problem 10. (Bonus) One of the twelve identically looking coins is fake. It is known that the fake coin differs in weight from the other eleven, but you don't know if it is heavier or lighter. How can you determine, in three weighings on a balance scale, which coin is fake? (Same as Example 2 but this time you have 12 coins instead of 9.)