Advanced Problem Solving II
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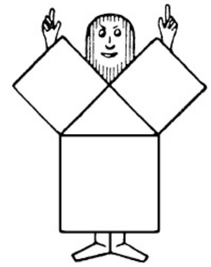

Homework 10
Pythagorean Triples

Problem 1. Regular n-gons are constructed on the legs a, b and a hypothenuse c of a right triangle. Check if the sum of the areas of the n-gons on the legs equals the area if the n-gon on the hypothenuse. For this, first deduce a formula for the area of a regular n-gon with side a.

Problem 2. Find all right integer triangles whose area equals half the perimeter. Make sure you showed that there are no other right integer triangles with this property.

Problem 3. Let (a, b, c) be a primitive Pythagorean triple where b is even. Show that b is divisible by 4. Prove this directly without using the theorem that describes all primitive Pythagorean triples.

Problem 4. Let (a, b, c) be a Pythagorean triple. Show that at least one of a and b is divisible by 3. Use this result and the result of the previous problem to prove that the area of an integer right triangle is an integer divisible by 6. Do not use the theorem that describes all primitive Pythagorean triples in this problem.

Problem 5. Let (a, b, c) be a Pythagorean triple. Show that at least one of a, b, c is divisible by 5.

Problem 6. Let $\triangle ABC$ be an integer right triangle. Show that its inradius is an integer. Use the Theorem that describes primitive Pythagorean triples.

Problem 7. Solve the previous problem using a different method: As in the previous problem, express the inradius in terms of the legs a and b and then use multiplication by the conjugate.

Problem 8. This problem is devoted to the search for Pythagorean triples whose legs differ in length by 1. Use the Theorem that describes primitive Pythagorean triples and write what the above condition means in terms of m and n. Show that this condition can be rewritten in the form $x^2 - 2y^2 = \pm 1$ (how to express x, y in terms of m, n?). Search for solutions to $x^2 - 2y^2 = \pm 1$ by consequently plugging in $x = 1, 2, 3$, etc. Find three such solutions and then recover corresponding Pythagorean triples whose legs differ in length by 1.

Problem 9. Show that the equation $x^2 - 2y^2 = \pm 5$ has no solutions with $\gcd(x, y) = 1$ by considering this equation modulo five. (What are the possible remainders when x^2 is divided by 5?) Explain how this implies that there are no primitive Pythagorean triples whose legs differ in length by 5.

Problem 10. Prove directly that there are no primitive Pythagorean triples whose legs differ in length by 5: Assume that $(a, a+5, c)$ is a primitive Pythagorean triple. This implies that $a^2 + (a+5)^2 = c^2$. Consider this equation modulo 5 to show that it has no integer solutions (a, c) with $\gcd(a, c) = 1$.