Advanced Problem Solving II
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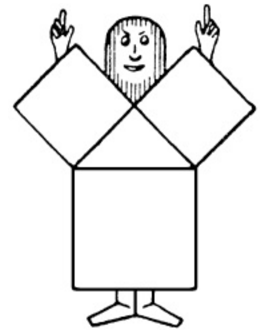
KSU, Fall 2017



Homework 11
Pythagorean Theorem: Number Theory and Geometry

Problem 1. Let (a, b, c) be a primitive Pythagorean triple with even b and odd a and c. Describe all such triples that satisfy $c = b + 1$. (The goal is to express b and c in terms of a). Use the obtained description to observe that for each odd a there exists a Pythagorean triple of the form $(a, b, b + 1)$. Find such a triple for $a = 37$.

Problem 2.

(a) Show that the sum of squares of two odd numbers cannot be equal to a square of an integer.
(b) Can the sum of squares of three odd numbers be equal to a square of an integer?

Problem 3.

(a) Show that a positive integer n of the form $n = 4k$ where k is a positive integer can always be written as a difference of squares of two nonnegative integers.
(b) Show that a positive integer n of the form $n = 4k + 1$ where k is a nonnegative integer can always be written as a difference of squares of two nonnegative integers.
(c) Show that a positive integer n of the form $n = 4k + 3$ where k is a nonnegative integer can always be written as a difference of squares of two nonnegative integers.

Problem 4. Show that positive integers n of the form $n = 4k + 2$ where k is a nonnegative integer cannot be written as a difference of squares of two nonnegative integers.

Problem 5. Show that a positive integer n of the form $n = 8k + 7$ where k is a nonnegative integer is not representable as a sum of three squares of nonnegative integers

Problem 6.

(a) Let $a \in \mathbb{Z}$. Prove that

$$a^2 \equiv 0,\ 1,\ \text{or}\ 4 \bmod 8$$

That is, prove that a square of an integer leaves a remainder of 0, 1, or 4 when divided by 8.

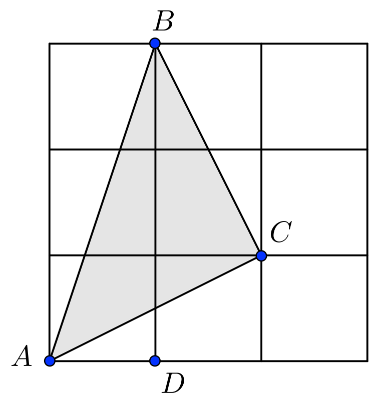
(b) Let m and n be natural numbers. Are there any perfect squares of the form

$$3^m + 3^n + 1?$$

Hint: Look at 3^{2k} and 3^{2k+1} modulo 8. What could $3^m + 3^n + 1$ be equal to modulo 8? In other words, what are the possible remainders after they are are divided by 8? What remainder can you get when you divide $3^m + 3^n + 1$ by 8? Now use part (a) of the problem.

Problem 7. Consider a triangle with the vertices $A(0,0)$, $B(1,3)$, and $C(2,1)$ as in the diagram below. Show that ABC is a $45° - 45° - 90°$ triangle. Let D be the point with the coordinates $(1,0)$. Show that $\tan \angle ABD = 1/3$ and $\tan \angle CBD = 1/2$. Explain why this proves the following trig identity:

$$\arctan 1/3 + \arctan 1/2 = \arctan 1.$$



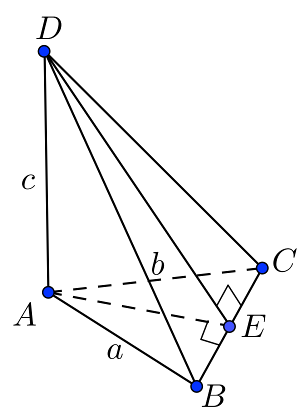
Problem 8. Use a method similar to the one described in the previous problem to show that

$$\arctan \frac{1}{2} = \arctan \frac{1}{3} + \arctan \frac{1}{7}.$$

Problem 9. Use the same method again to show that

$$\arctan 1 + \arctan 2 + \arctan 3 = 180°.$$

Problem 10. Consider a right pyramid $ABCD$ with $\angle BAC = \angle CAD = \angle DAB = 90°$. Show that the sum of squares of the areas of ABC, ABD, and ACD is equal to the square of the area of BCD.



That is, show that

$$S_{ABC}^2 + S_{ABD}^2 + S_{ACD}^2 = S_{BCD}^2.$$

Use the fact that the altitudes from vertices A and D to the side BC have a common foot E. Also, since AD is perpendicular to two lines in the base that pass through A, it is perpendicular to every line in the base, in particular $\angle DAE = 90^\circ$.