Advanced Problem Solving II
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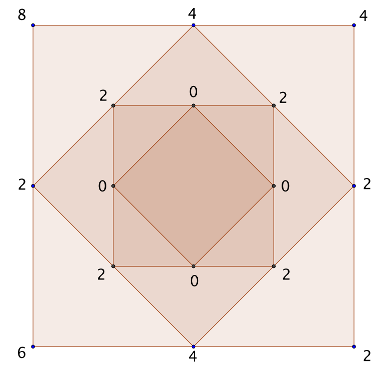

The Four Numbers Game

Problem 1. (a) Show that the game (A, B, A, B) ends in at most two steps.
(b) Play the game
$$(3.2, \pi, \frac{22}{7}, 3.14).$$
Do not use approximations. Do not use any further results.

Problem 2. (a) Show that the games (na, nb, nc, nd) and (a, b, c, d) are of the same length.
(b) Show that $L(a, b, c, d) = L(a+e, b+e, c+e, d+e)$.
(c) Show that $L(a, b, c, d) = L(na+e, nb+e, c+e, nd+e)$. (We will say that games (a, b, c, d) and $(na+e, nb+e, c+e, nd+e)$ are *equivalent*.)

Problem 3. Show that the possible lengths of 3-Numbers Games, played with non-negative integers, are 0, 1, and ∞. Show that no other lengths are possible and give examples of games of length 0, 1, and ∞. Hint: Go backwards starting with $(0, 0, 0)$ or consider possible combinations of parities in the game (a, b, c).

Problem 4. (a) Show that $L(a, b, b, a) \leq 3$.
(b) Show that $L(a, b, a, c) \leq 4$. (Note that by rotation we can conclude that the same statement applies to the game (b, a, c, a).)

Problem 5. Show that a game (a, b, c, d), where $a \geq c \geq b \geq d$ are non-negative integers, has a length of at most 4.

Problem 6. Show that a game (a, b, c, d), where $a \geq b \geq d \geq c$ are non-negative integers, has a length of at most 6. (Hint: Use Problem 4.)

Problem 7. Consider an 8-Numbers Game (a, b, c, d, e, f, g, h) where all the numbers are integers. Show that all the numbers appearing from step eight onward are even.

Problem 8. Use previous problem to show that every 8-Numbers Game played with non-negative integers has finite length. More precisely, use induction to show that if A is the largest of the eight integers in the beginning of the game and k is the least integer such that $A < 2^k$, then the length of the game is at most $8k$.

Problem 9. Recall that a game (a, b, c, d) with $a \geq b \geq c \geq d \geq 0$ is called additive if $a = b + c + d$. Construct an additive game S which is equivalent to $(14, 5, 3, 1)$. Next, find a game $T = (a, b, c, d)$ with $a \geq b \geq c \geq d$ that turns into S in one step.

Problem 10. Given an additive game (a, b, c, d) with $a = b+c+d$ construct a game (x, y, z, w) that turns into (a, b, c, d) in one step.

Problem 11. Given a game $S = (a, b, c, d)$ with $a > b + c + d$ and $a, b, c, d \geq 0$, construct an additive game equivalent to S.

Problem 12. Show that for any non-negative number N there is an 8-Numbers Game of length N. For this, start with a 4-Numbers Game (a, b, c, d) and construct an 8-Numbers Game based on (a, b, c, d) of the same length. Next, use the corresponding result for 4-Numbers Games discussed in class.