Advanced Problem Solving II
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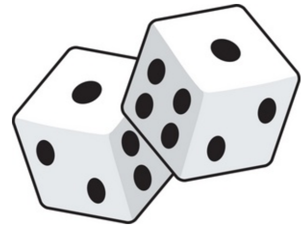

Generating Functions
Homework 2

Problem 1. Find the coefficient of (a) x^3, (b) x^8, (c) x^{10}, and (d) x^{15} in the expansion of $(1+x^3+x^5)^{15}$.

Problem 2. Find the generating function $f(x)$ for the standard tetrahedral die labeled 1, 2, 3, and 4. Find the probability distribution for the sum of two standard tetrahedral dice, that is, find the probabilities for the sum of rolled values to be $1, 2, \dots, 8$. Next, expand $f(x)^2$ and explain the relation between the coefficients in the expansion and the probabilities that you just found.

Problem 3. Let $f(x) = c_n x^n + \cdots + c_0$ be a polynomial with integer coefficients. Show that if a rational number a/b, written in lowest terms, is a root of $f(x)$ then a divides c_0 and b divides c_n. Use this statement to show that $f(x) = x^3 - 2x + 3$ has no rational roots.

Problem 4. Let $f(x)$ be a polynomial of degree 2 or 3. Show that if $f(x)$ is reducible over $\mathbb{Q}$, then it has a rational root. (A polynomial with rational coefficients is reducible over $\mathbb{Q}$ if it can be written as a product of two polynomials of smaller degrees with coefficients in $\mathbb{Q}$. For example, $x^3 - 1 = (x-1)(x^2+x+1)$, so it is reducible over $\mathbb{Q}$.)

Problem 5. Let $f(x)$ be the generating function for the standard tetrahedral die with 1, 2, 3, 4 on its faces. Write $f(x)^2$ as a product of irreducible factors over $\mathbb{Q}$.

Problem 6. Let $g(x)$ be the generating function for a tetrahedral die labeled with some positive integers. Explain why $g(0) = 0$ and $g(1) = 4$.

Problem 7. Let $g(x)$ and $h(x)$ be generating functions for two tetrahedral dice (not necessarily the same) labeled with positive integers. Given that the probability distribution for the sum of these two dice is the same as that for two standard tetrahedral dice, find $g(x)$ and $h(x)$ and then explain how the initial dice are labeled. Find all possible solutions. Use generating functions.

Problem 8. How to label a tetrahedral die and a 9-sided die so that that the probability distribution for the sum of these two dice is the same as that for two standard 6-sided dice? Use generating functions.

Problem 9. Prove that

$$1 + x + \cdots + x^d = \frac{1 - x^{d+1}}{1 - x}.$$

Explain how this implies that for $|x| < 1$

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Problem 10. Use the result of the previous problem to come up with a formula for

$$1 + x^a + x^{2a} + x^{3a} + \cdots$$

Use this formula to expand $\frac{1}{x^3-1}$ and $\frac{1}{8-x^2}$.

Problem 11. If we have two denominations of postage stamps, 5 cents and 6 stamps, what is the highest postal (integer) rate that we cannot pay? Do not use any results from class. Do this by hand (no generating functions) and explain why your answer cannot be paid, but any higher rate can.

Problem 12. Repeat our argument from class for $a = 5$ and $b = 6$ to show that the smallest number that cannot be paid is $5 \cdot 6 - 5 - 6 = 19$. Do not just use the formula, repeat the argument making use of generating functions.