Advanced Problem Solving II
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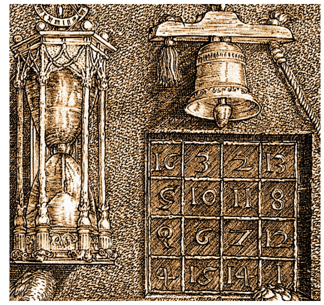
KSU, Fall 2017



Solutions 4
Magic Squares

A *Magic Square* is an $n \times n$ table of numbers from 1 to n^2 such that the sum of the entries in every row, every column, and the two diagonals is the same.

Problem 1. What is the magic sum (the sum of the entries in every row, column, or diagonal) in a 3×3, 4×4 magic square? What is the magic sum in an $n \times n$ magic square? (Hint: What is the overall sum of all the entries?)

Problem 2. Show that there is only one 3×3 magic square up to rotations and reflections. (Hint: where can 1 be placed?)

Problem 3. Is it possible to complete the following table to a magic square? If not, prove that it is impossible. If it is possible, find all such ways an explain why no other such magic squares exist.

2	3		
4			

Problem 4. Is it possible to complete the following table to a magic square? If not, prove that it is impossible. If it is possible, find all such ways an explain why no other such magic squares exist.

16			
		11	8
4			1

Problem 5. Let S be an n by n magic square. Show that if you replace every integer a in S with $n^2 + 1 - a$, you again get a magic square.

Problem 6. A 3 by 3 magic square can be constructed in the following way. We will fill in the table writing numbers in increasing order. Start with a 1 in the middle top square. At each next step we will be writing next number in the square that is diagonally up and to the right from the current square. If we get outside the 3 by 3 square we reduce both coordinates modulo 3. For example, 2 would have to be placed in the square $(3,4)$, so it reduces to $(3,1)$, the lower right square. Next, 3 would have to be placed into $(4,2)$, which reduces to $(1,2)$. Whenever the square that is diagonally up and to the right from the current square is taken, we move to the square which is directly under the current square. For example, 4 cannot be placed into $(2,3)$, so we put it in $(1,1)$. Use this method to construct a 5×5 and a 7×7 magic square.

Problem 7. Construct a 6×6 magic square using the following steps:

(1) Fill out the square with four identical 3×3 magic squares (four quadrants):

I	II
III	IV

(2) Add 18 to each entry in quadrant II, 27 to each entry in quadrant III, and 9 to each entry in quadrant IV. Check that in the resulting 6×6 square all the entries are different (i.e. numbers from 1 to 36) and the sum in each column is 111.

(3) Swap a few entries between quadrants I and III in the first two columns to obtain a 6×6 magic square.

Problem 8. A magic cube is an $n \times n \times n$ table filled with numbers from 1 to n^3, each used exactly once, so that the sum of the entries in all rows, columns, pillars, diagonals in every layer, and the four space diagonals is the same. Does there exist a $2 \times 2 \times 2$ magic cube? What about a $3 \times 3 \times 3$ one?

Problem 9. Compared to the definition in the previous problem, drop the requirement about diagonals in every layer. That is, we now call an $n \times n \times n$ cube magic if it is filled with numbers from 1 to n^3, each used exactly once, so that the sum of the entries in all rows, columns, pillars, and the four space diagonals is the same. Does there exist such a $2 \times 2 \times 2$ magic cube?

Problem 10. We say that a 3 by 3 table of integers 1 through 9 is an almost magic square if each of its row, column, and diagonal sums equals 14, 15, or 16. Show that such a square can only have 4, 5, or 6 in its center. Give examples of almost magic 3 by 3 squares with 4, 5, and 6 in the center. Make sure that your example with 5 in the center is almost magic, but not magic.