Advanced Problem Solving II
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Homework 5
Stable Marriages

Problem 1. (a) Given n boys and n girls, how many ways are there to break them into n pairs?

(b) Given $2n$ girls, how many ways are there to break them into n pairs?

Problem 2. You wish to assign four girls, A, B, C, and D, to two rooms, two per room. Come up with preference lists for the girls, where each of them ranks the remaining three as possible roommates (no ties allowed), such that no stable assignment exists.

Problem 3. Let A, B be two girls and α, β two boys. Come up with preference lists such that both possible pairings are stable.

Problem 4. Show that Gale-Shapley algorithm for pairing up n boys and n girls takes at most $n^2 - n + 1$ rounds. (Hint: In each round at least one boy goes down his list. What happens in the first round?)

Problem 5. For the following ranking matrix

	A	B	C	D
α	1,4	2,3	3,1	4,2
β	4,1	2,1	3,3	1,3
γ	1,3	2,4	3,4	4,1
δ	4,2	3,2	2,2	1,4

find all the feasible partners for α and then determine $\alpha's$ optimal and pessimal partners. (Do not run the Gale-Shapley algorithm here.)

Problem 6. Run the Gale-Shapley algorithm for the ranking matrix below twice. First time let the boys ($\alpha, \beta, \gamma, \delta$) propose, and then let the girls (A, B, C, D) propose.

	A	B	C	D
α	1,3	2,3	3,2	4,3
β	1,4	4,1	3,3	2,2
γ	2,2	1,4	3,4	4,1
δ	4,1	2,2	3,1	1,4

Conclude that there is only one stable pairing for this ranking matrix. (Use the theorem that the Gale-Shapley algorithm where boys propose is male-optimal and female-pessimal.)

Problem 7. Come up with a 3×3 ranking matrix for which Gale-Shapley algorithm takes 5 rounds.

Problem 8. Show that in the Gale Shapley algorithm

(a) If a boy proposes to his last choice, then this may only happen in the last round;

(b) At most one boy may end up with his last choice.

Problem 9. Use previous problem to conclude that the Gale-Shapley algorithm takes at most $n^2 - 2n + 2$ rounds.

Problem 10. For the following ranking matrix

$$\begin{bmatrix} 1,n & 2,n-1 & 3,n-2 & \dots & n,1 \\ n,1 & 1,n & 2,n-1 & \dots & n-1,2 \\ n-1,2 & n,1 & 1,n & \dots & n-2,3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2,n-1 & 3,n-2 & 4,n-3 & \dots & 1,n \end{bmatrix}$$

consider the pairing where each girl gets her kth choice for some fixed k with $1 \leq k \leq n$. Show that such a paring is stable. Hint: consider two pairs (A, α) and (B, β) where the husbands are kth choices of their wives and show that A and β would not run away. What are the numbers of the wives on their husbands' lists? It is useful to notice that for each entry i, j in the ranking matrix we have $i + j = n + 1$.