Advanced Problem Solving II
Jenya Soprunova
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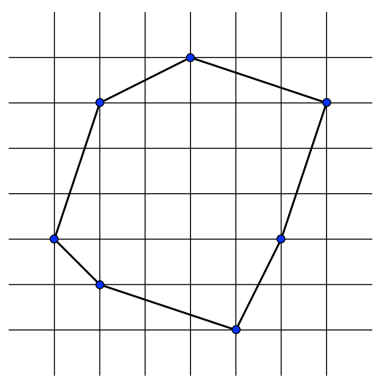
Pick's Formula

Let's call a point in the plane a *lattice* (or an *integer*) point if both of its coordinates are integers. Let P be a *lattice* polygon in the plane, that is, a polygon all of whose vertices are lattice points. Let I be the number of lattice points that are strictly inside P and B be the number of lattice points that are on the boundary of P. The goal of this homework assignment is to prove Pick's formula which expresses the area A of P in terms of I and B:

$$A = I + \frac{B}{2} - 1.$$

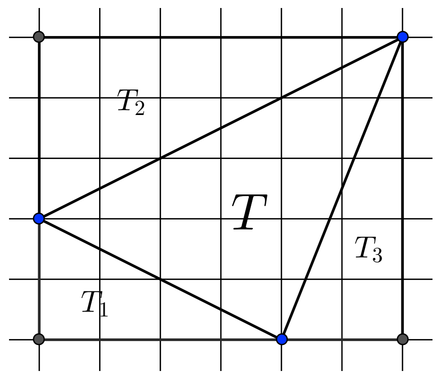
Problem 1. For the polygon below, compute I, B, and A directly, not using Pick's formula. Check that Pick's formula is satisfied.



Problem 2. Let P be a lattice rectangle with the sides parallel to the coordinate axes. We can then assume that the vertices of P are at the points $(0,0)$, $(a,0)$, $(0,b)$, and (a,b) for some integers a and b. Check that Pick's formula holds for P.

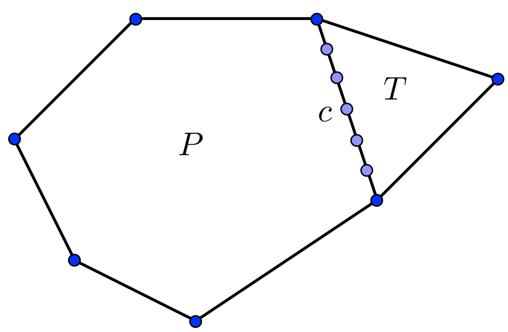
Problem 3. Now let T be a lattice triangle with two sides parallel to the coordinate axes. We can then assume that the vertices of T are at the points $(0,0)$, $(a,0)$, and $(0,b)$ for some integers a and b. Check that Pick's formula holds for T. Follow the plan: Consider a rectangle P from previous problem. Let I_P, B_P and A_P be the numbers of interior lattice points, boundary lattice points, and the area for P, while I_T, B_T, and A_T be the corresponding parameters for T. Let c be the number of lattice points on the hypothenuse of T (not counting the vertices). From Problem 2, we already know that $A_P = I_P + B_P/2 - 1$. Express A_P, I_P, and B_P in terms of I_T, B_T, A_T, and c. Plug into Pick's formula for P. Obtain Pick's formula for T.

Problem 4. Next, let T be an arbitrary lattice triangle. Check that Pick's formula holds for T. Follow the plan: Consider a rectangle P whose sides are parallel to the coordinate axes, so that P shares one of the vertices with T and two other vertices of T are on the sides of P. Notice that P is broken into four triangles, T, T_1, T_2, and T_3, where the last three are all of the kind considered in Problem 3.



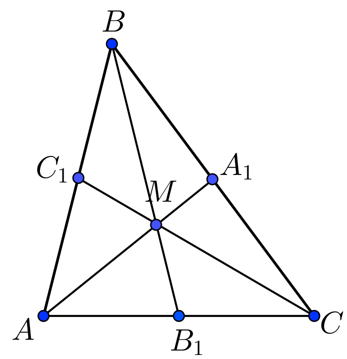
Let A, I, B be the area and the numbers of the interior and boundary lattice points for T and A_1, I_1, B_1, A_2, I_2, B_2, A_3, I_3, B_3 be the corresponding parameters for T_1, T_2, and T_3. We already know that Pick's formula holds for P and the three triangles. Expressing the parameters for P in terms of the parameters for the triangles, prove Pick's formula for T.

Problem 5. Assume Pick's formula holds for a polygon P. Show that it holds for the polygon $P \cup T$, where T is a triangle and P and T share a side. Hint: This is very similar to Problem 2. Denote the number of lattice points on the common side by c.



Problem 6. Use the result from previous problem and the fact that any lattice polygon P (not necessarily convex) can be broken into finitely many lattice triangles, so that any two of them either do not overlap or share a side, to conclude that Pick's formula holds for any lattice polygon .

Problem 7. Let M be the centroid of a triangle ABC . Show that if you connect M to the vertices of ABC you will break ABC into three triangles of equal area. Show that no other point N inside ABC has this property. This problem does not require Pick's formula. Use some simple plane geometry here.



Problem 8. Let ABC be a lattice triangle that has just one integer point inside and the only lattice points on its boundary are the vertices. Show that this interior lattice point is the centroid of ABC. Hint: Use Pick's formula ($A = I + B/2 - 1$) and the previous problem.

Problem 9. Pick's formula, in particular, says, that in order for a polygon to have a large area, it needs to have a large overall number of lattice points inside and on the boundary. Show that this is not the case in 3D, that is, a 3D polytope with fixed I and B can have a huge volume. For this, consider the tetrahedron T with the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,c)$, where c is a positive integer.

Problem 10. Let P be a lattice polygon with k polygonal lattice holes (that is, k lattice polygons were cut out of P). Prove the following version of the Pick's formula

$$A_P = I_P + \frac{B_P}{2} + k - 1,$$

where A_P is the area of P (with area of the holes subtracted), I_P is the number of interior lattice points (not counting the ones in the holes), and B_P is the number of lattice points on the boundary of P. Note that the boundary of P consists of the exterior boundary and the interior boundary, where the holes were cut out. That is, B_P counts the number of lattice points on the boundary of P, including the interior boundary. In the diagram below, that provides an example of a lattice polygon with 3 polygonal lattice holes, we have $B_P = 27$.

Hint: Use Pick's formula for each of the holes and for the initial polygon before the holes were cut out.