Advanced Problem Solving II
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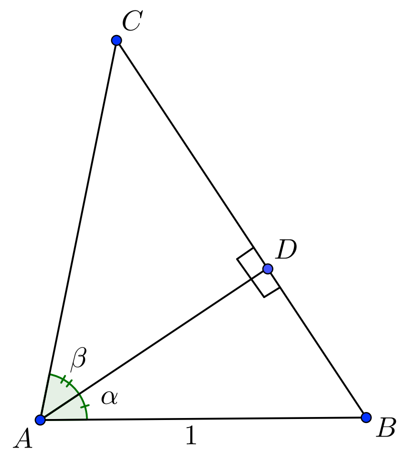
Lattice Point Geometry

Problem 1. Prove the trig identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Follow the plan: Let ABC be a triangle with an altitude AD. Assume $AB = 1$, $\angle BAD = \alpha$, and $\angle CAD = \beta$, as in the diagram below. Find the lengths of AD, AC, CD, and BD in terms of α and β. Express the area of ABC in two different ways and deduce the formula.



Problem 2. Use trig identities to show that

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Problem 3. Use induction and the previous problem to show that

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

Problem 4. Deduce an expression for $\tan n\theta$ in terms of $\tan\theta$ for even n. Follow the plan used in class for odd n.

Problem 5. Let $\theta = 2\pi/7$. Prove that $\tan\theta$ is irrational by deducing first a formula for $\tan 7\theta$ is terms of $\tan\theta$ and using the rational root test. Explain why this implies that there is no regular lattice 7-gon in the plane.

Problem 6. Show that there exists an equilateral lattice triangle in the 3-space.

Problem 7. Give an example of an equilateral convex lattice hexagon in the plane. (We proved in class that there are no regular lattice hexagons, so in your example the hexagon will not be equiangular.)

Problem 8. Give an example of (a) an equilateral convex lattice octagon in the plane and (b) an equilateral convex lattice decagon in the plane.

Problem 9. Does there exist a lattice segment of length $\sqrt{15}$?

Problem 10. Does there exist a lattice triangle with the sides $\sqrt{10}, \sqrt{10}, 2\sqrt{5}$? What about a triangle with sides $\sqrt{10}, \sqrt{10}, \sqrt{13}$? (Hint: We proved in class that tangent of a lattice angle is rational.)

Problem 11. Find all the lattice triangles with two sides equal to $\sqrt{5}$. For this, draw all possible lattice segments of length $\sqrt{5}$ that start at the origin. How can one put together two such segments to get a triangle? What are the possible values for the third side?

Problem 12. Show that there are no equilateral lattice pentagons in the plane. Follow the plan: Assume such a pentagon exists and consider a smallest such pentagon (that is, a pentagon which is not a dilate of another lattice pentagon). Let $(a_1, b_1), \dots, (a_5, b_5)$ be the integer vectors along the sides of the pentagon. Then the sum of these five vectors equals zero, that is, $a_1 + a_2 + a_3 + a_4 + a_5 = 0$ and $b_1 + b_2 + b_3 + b_4 + b_5 = 0$. Since all the sides are equal, we have

$$l^2 = a_1^2 + b_1^2 = \cdots = a_5^2 + b_5^2,$$

where l is the length of a side of the pentagon. (Note that while l^2 is an integer, l could be irrational.) The idea is to keep track of the parities of the a_i's and b_i's. All of them cannot be even, as if this is the case there is a smaller equilateral pentagon. Assume there is a vector (a_i, b_i) with both components even. What does this imply for l^2 and other vectors? Can you rule this case out? Next, assume there is a vector (a_i, b_i) with both components odd. What does this imply for l^2 and other vectors? Can you rule this case out? Finally, consider the case where each pair has an even and an odd component. How can this go together with $a_1 + a_2 + a_3 + a_4 + a_5 = 0$ and $b_1 + b_2 + b_3 + b_4 + b_5 = 0$?