Advanced Problem Solving II
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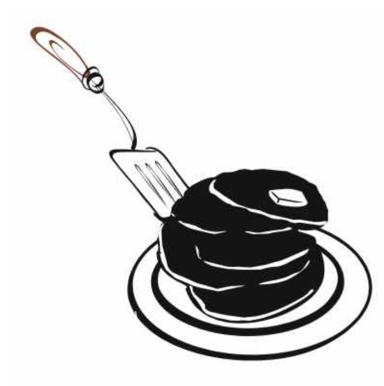
KSU, Fall 2017



Sorting Pancakes

You are given a stack of n pancakes of sizes 1 through n in arbitary order. You can insert a spatula in the stack and then flip the stack of pancakes on top of the spatula. That is, you can lift all the pancakes above the spatula and replace them in reverse orburntder. This operation is called a *spatula flip*. In many of the problems bellow the goal is to use spatula flips to sort the stack so that the pancakes decrease in size from bottom to top, in which case we will say that the pancakes are sorted.

Example 1. A stack of three pancakes with the largest one in the middle and the smallest one at the bottom can be sorted in two steps, that is, using two spatula flips. This can be recorded as follows:

$$\begin{bmatrix}2\\3\\1\end{bmatrix} \xrightarrow{s_2} \begin{bmatrix}3\\2\\1\end{bmatrix} \xrightarrow{s_3} \begin{bmatrix}1\\2\\3\end{bmatrix},$$

where s_2 indicates that we flipped two pancakes and s_3 indicates that we flipped three.

In some of the questions, the pancakes are burnt on one side and the goal is not only to sort them according to sizes but to also make sure that the burnt sides are facing downward. To show that a pancake is placed with its burnt side up we will be using a negative sign in front of the number indicating the pancake.

Example 2. A stack $\begin{bmatrix}2\\-3\\1\end{bmatrix}$ can be sorted using four flips. Indeed, we can perform the following sequence of flips

$$\begin{bmatrix}2\\-3\\1\end{bmatrix} \xrightarrow{s_2} \begin{bmatrix}3\\-2\\1\end{bmatrix} \xrightarrow{s_1} \begin{bmatrix}-3\\-2\\1\end{bmatrix} \xrightarrow{s_3} \begin{bmatrix}-1\\2\\3\end{bmatrix} \xrightarrow{s_1} \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

Homework Problems

Problem 1. Draw all possible stacks of three pancakes of sizes 1, 2, and 3. Connect two stacks with an edge if you can get from one stack to another in one flip. Mark each edge with the corresponding flip (s_2 or s_3). Looking at this graph, find $f(3)$, the smallest number of flips needed to sort a pile of three pancakes in the worst-case scenario.

Problem 2. Show that the stack of Example 2 cannot be sorted in fewer than 4 flips. For this, carefully examine all possible 3-flip scenarios.

Problem 3. Here is a way to sort a particular stack of 5 (unburnt) pancakes:

$$\begin{bmatrix}1\\5\\3\\4\\2\end{bmatrix} \to \begin{bmatrix}5\\1\\3\\4\\2\end{bmatrix} \to \begin{bmatrix}2\\4\\3\\1\\5\end{bmatrix} \to \begin{bmatrix}4\\2\\3\\1\\5\end{bmatrix} \to \begin{bmatrix}1\\3\\2\\4\\5\end{bmatrix} \to \begin{bmatrix}3\\1\\2\\4\\5\end{bmatrix} \to \begin{bmatrix}2\\1\\3\\4\\5\end{bmatrix} \to \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}.$$

Based on this example, explain a general algorithm for sorting n unburnt pancakes. Show that using this algorithm one can sort n pancakes using $2n-3$ spatula flips, that is, $f(n) \leq 2n-3$.

Problem 4. Based on the previous problem, we know that $f(4) \leq 2 \cdot 4 - 3 = 5$. Prove that in fact $f(4) = 4$, that is, the above algorithm is not optimal for $n = 4$. For this, you need to see that any stack of four pancakes can be sorted in fewer than 4 steps and that at least one stack cannot be sorted in 3 steps.

Problem 5. Draw all possible stacks of two burnt pancakes of sizes 1 and 2. Connect two stacks with an edge if you can get from one stack to another in one flip. Mark each edge with the corresponding flip (s_1 or s_2). Looking at this graph, find $g(2)$, the smallest number of flips needed to sort a pile of two burnt pancakes in the worst-case scenario.

Problem 6. Here is a way to sort a particular stack of 4 burnt pancakes:

$$\begin{bmatrix}-3\\2\\-4\\1\end{bmatrix} \to \begin{bmatrix}4\\-2\\3\\1\end{bmatrix} \to \begin{bmatrix}-4\\-2\\3\\1\end{bmatrix} \to \begin{bmatrix}-1\\-3\\2\\4\end{bmatrix} \to \begin{bmatrix}3\\1\\2\\4\end{bmatrix} \to \begin{bmatrix}-3\\1\\2\\4\end{bmatrix} \to \begin{bmatrix}-2\\-1\\3\\4\end{bmatrix} \to \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

Based on this example, explain a general algorithm for sorting n burnt pancakes. Show that using this algorithm one can sort n pancakes using $3n$ spatula flips, that is, $g(n) \leq 3n$. Compare with previous problem and observe that this algorithm is not optimal for $n = 2$.

Note that the algorithm of Problem 6 can be improved to only require $2n$ steps. Furthermore, one can make a further improvement and show that for $n \geq 10$ we have $g(n) \leq 2n-2$, but even this is only a bound. For example, $g(11) = 19$, while the formula gives a bound $g(11) \leq 20$.

We next consider the case of n unburnt pancakes that come in two sizes only, namely, 1 and 2. The goal is to sort a stack of such pancakes so that the larger pancakes are at the bottom and the smaller ones are at the top.

Problem 7. Come up with a way of sorting a stack of n unburnt pancakes that come in two sizes in $n-1$ spatula flips. (Hint: Use induction here. If you have two pancakes of the same size next to each other then you can "glue" them as you will never need to separate them and consider this stack as a stack of $n-1$ pancakes. If you have a larger pancake at the bottom you can "glue" it to the table. In what case you cannot make any reductions of this kind?)

Problem 8. Let $h(n)$ be the smallest number of flips needed in the worst case scenario to sort n unburnt pancakes of size 1 and 2. In the previous problem we proved that $g(n) \leq n-1$. Show that $g(n) = n-1$ by proving that the stack $\begin{bmatrix} \vdots \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ cannot be sorted in fewer than $n-1$ flips. (Hint: Observe that in the resulting stack we only have one larger pancake next to a smaller one. How many do we have in the initial stack? This tells us that we will need to insert a spatula between pancakes in all such pairs except, possibly, one of them. We will also need to insert a spatula under the small pancake at the bottom.)

We next consider a similar problem where all pancakes in a stack are of the same size, but all of them are burnt on one side. We would like to sort such a pile so that all pancakes are burnt side down. Let $u(n)$ be the smallest number of flips needed to sort such a stack of n pancakes in the worst case scenario. Note that since the pancakes are all the same size we can use pluses and minuses to denote pancakes in a stack.

Problem 9. Show that $u(n) \leq n$. (This is very similar to Problem 7.)

Problem 10. Show that $u(n) = n$. (This is very similar to Problem 8.)