Geometry for HS Teachers
Fall 2011

Dissections

Definition 0.1. Two convex polygons P and Q are said to be *congruent by dissection* if you can cut P into convex polygons, rearrange the obtained pieces (it's allowed to turn the pieces over), put them together with no overlaps and obtain a polygon congruent to Q.

The goal of this sequence of problems is to prove the Bolyai-Gerwien Theorem that says that any two convex polygons of the same area are congruent by dissection.

Problem 1. Let P and Q be convex polygons. And S be a square. Given that P and S are congruent by dissection, and Q and S are also congruent by dissection. Show that this implies that P and Q are congruent by dissection.

Problem 2. Show that any convex polygon can be cut into triangles.

Problem 3. Cut a triangle into parts that can be rearranged to get a rectangle. (It's okay to turn the parts over.) Explain how this works for a general triangle.

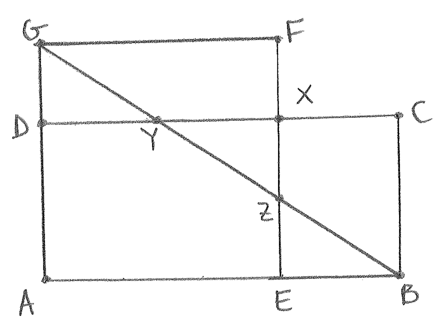
Next, we need to show that any rectangle is congruent by dissection to some square. This is broken into two problems.

Problem 4. Let P be a rectangle. Show that it is congruent by dissection to a rectangle with sides a and b such that $a \leq b < 4a$.

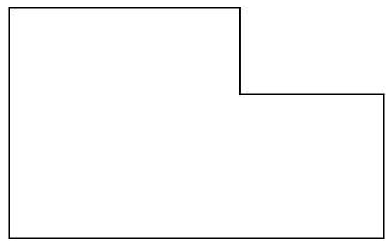
Problem 5. Let P be a rectangle with sides a and b such that $a \leq b < 4a$. Show that it is congruent by dissection to a square.

Plan: Let $ABCD$ be the rectangle and $AEFG$ be the square of the same area (see the diagram). Let $AB = b$, $AD = a$. Let X be the point of intersection of DC and FE. Draw a segment that connects B and G. Let the points of intersection of BG with DC and FE be Y and Z correspondingly.

Then $AE = AG = \sqrt{ab}$ (why?). Show that $GD = ZE$. Explain why this implies that the triangles GDY and ZEB are congruent. Next, show that the triangles GFZ and YCB are congruent. Use this information to cut $ABCD$ into convex pieces, rearrange them and get $AEFG$. Find the length of YX and show that $YX > 0$ if and only if $b < 4a$.



Problem 6. Consider the following shape which is a union of two squares.



Cut this shape into three pieces such that they can be rearranged to get a square. If the first square has side a and the second has side b, what is the side of the new square?

Problem 7. Prove the Bolyai-Gerwien Theorem: Any two convex polygons of the same area are congruent by dissection.