

Filling boxes with bricks

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Brickwork is usually made of bricks whose measurements are $1 \times 2 \times 4$, so that they can fit together in many positions. It is very remarkable, however, that this flexibility does not help at all when it is required to fill a given rectangular box entirely with such bricks. It can be proved (and a more general result will be proved in this note) that if the box cannot be filled trivially (i.e., with all bricks in parallel position), then it cannot be filled at all. For instance, if the box is $10 \times 10 \times 10$, it cannot be filled trivially with $1 \times 2 \times 4$ bricks, since 4 does not divide 10. By the theorem just mentioned, it cannot be filled at all without breaking the bricks, although the number of bricks that is needed is an integer (viz., 125).

It has to be remarked that the situation is different with bricks with measurements $1 \times 2 \times 3$. For, the flat box $1 \times 5 \times 6$ can be filled (see figure), but it cannot be filled trivially.

In this note we shall determine all bricks which share the above-mentioned property of the $1 \times 2 \times 4$. The problem will be solved for an arbitrary number of dimensions, and the answer will be quite simple.

The decisive idea of this paper is a generalization of a trick occurring in the solution of the following well-known problem. Cut out two diagonally opposite squares of an 8×8 checker-board, and show that it is impossible to cover the remaining 62 squares by 31 dominoes of size 1×2 . The impossibility is shown as follows. Each domino will cover both a black and a white square. On the other hand, the region to be covered does not have the same number of squares of each colour since the two removed squares had the same colour.

Measurements of n -dimensional bricks and boxes will be expressed by integers. We consider bricks $a_1 \times \cdots \times a_n$ and a box $A_1 \times \cdots \times A_n$. The box $A_1 \times \cdots \times A_n$ is called a *multiple* of the brick $a_1 \times \cdots \times a_n$, if there are integers q_1, \dots, q_n such that the numbers $q_1 a_1, \dots, q_n a_n$ form a rearrangement of $A_1 \times \cdots \times A_n$. (Example: $10 \times 16 \times 17$ is a multiple of $1 \times 2 \times 4$.) It is easily seen that the box can be filled trivially if and only if it is a multiple of the brick.

The brick $a_1 \times \cdots \times a_n$ is called *harmonic* if the numbers a_1, \dots, a_n can be rearranged into a'_1, \dots, a'_n with $a'_1 | a'_2, a'_2 | a'_3, \dots, a'_{n-1} | a'_n$. (Example: the brick $2 \times 1 \times 6 \times 6$ is harmonic.)

The purpose of this note is to show that the harmonic bricks share the above-mentioned property of the $1 \times 2 \times 4$ bricks (Theorem 2), and that the nonharmonic bricks do not have this property (Theorem 3). In other words, if the brick is not harmonic, then there are boxes which can be filled, but cannot be filled trivially. If the brick is harmonic, then every box that can be filled, can already be filled trivially.

We first prove

THEOREM 1. If the box $A_1 \times \cdots \times A_n$ can be filled with bricks $a_1 \times \cdots \times a_n$, then at least one of the A_i is a multiple of a_1 , at least one of the A_i is a multiple of a_2 , etc. (this does not necessarily imply that the box is a multiple of the brick; cf. the case of the box $1 \times 5 \times 6$ and the brick $1 \times 2 \times 3$).

Proof. It suffices to consider a_1 . As any brick $a_1 \times \cdots \times a_n$ can be subdivided into brick $a_1 \times 1 \times \cdots \times 1$, we may start from the assumption that the box is filled, in some way or another, with such bricks $a_1 \times 1 \times \cdots \times 1$. We divide each one of them into a_1 cubes $1 \times \cdots \times 1$. The box now contains $A_1 \cdots A_n$ cubes. We give these cubes coordinates (k_1, \dots, k_n) ($1 \leq k_1 \leq A_1, \dots, 1 \leq k_n \leq A_n$). We consider the sum

$$S(A) = \sum_{k=1}^A e^{2\pi i k/a_1},$$

and the multiple sum

$$\sum_{k_1=1}^{A_1} \cdots \sum_{k_n=1}^{A_n} e^{2\pi i (k_1 + \cdots + k_n)/a_1} = S(A_1) \cdots S(A_n).$$

Each term in this multiple sum corresponds to a cube in the box. These terms can be grouped together in blocks of a_1 terms each, combining terms corresponding to cubes belonging to one and the same brick. In such a group of terms, one of the indices runs through a set of a_1 consecutive integers, and the other indices remain constant. It follows that the contribution of such a group is zero, irrespective of the orientation of the brick.

We infer that the multiple sum over all cubes vanishes and therefore one of the $S(A_j)$ equals zero. Since

$$S(A_j) = x + x^2 + \cdots + x^{A_j} = x(x^{A_j} - 1)/(x - 1)$$

with $x = e^{2\pi i/a_1}$, we have $e^{2\pi i A_j/a_1} = 1$. It follows that A_j is a multiple of a_1 .

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THEOREM 2. If a box is filled with harmonic bricks $a_1 \times \cdots \times a_n$, then the box is a multiple of the brick.

Proof. We use induction with respect to n . If $n = 1$ we have a triviality. Assuming that the assertion holds in the $n - 1$ -dimensional case, we shall tackle the case of n dimensions.

Without loss of generality we may assume that $a_1|a_2, a_2|a_3, \dots, a_{n-1}|a_n$. If the box is $A_1 \times \cdots \times A_n$, we have, by the previous theorem, that one of the A_i is a multiple of a_n . Assume that A_n is a multiple of a_n .

We consider an $(n - 1)$ -dimensional face $A_1 \times \cdots \times A_{n-1}$. This is entirely filled with $(n - 1)$ -dimensional bricks of various sizes, viz., $a_2 \times \cdots \times a_n$, or $a_1 \times a_3 \times \cdots \times a_n, \dots$, or $a_1 \times a_2 \times \cdots \times a_{n-1}$. Each one of these can be subdivided into $(n - 1)$ -dimensional bricks $a_1 \times \cdots \times a_{n-1}$, by virtue of the divisibility property of a_1, \dots, a_n . The brick $a_1 \times \cdots \times a_{n-1}$ being harmonic, we observe that $A_1 \times \cdots \times A_{n-1}$ is a multiple of $a_1 \times \cdots \times a_{n-1}$ by virtue of the induction hypothesis. Since we know that $a_n|A_n$, we infer that $A_1 \times \cdots \times A_n$ is a multiple of $a_1 \times \cdots \times a_n$.

THEOREM 3. If the brick $a_1 \times \cdots \times a_n$ is not harmonic, then there is a box which can be filled without being a multiple of the brick.

Proof. We may assume that $n > 1$, $a_1 \leq a_2 \leq \cdots \leq a_n$, and that k is the largest integer for which a_{k-1} does not divide a_k (so $2 \leq k \leq n$).

A box $(a + b) \times ab$ can be filled with bricks $a \times b$ (see figure, where $a = 2, b = 3$). Therefore the box

$$a_1 \times \cdots \times a_{k-2} \times (a_{k-1} + a_k) \times a_{k-1}a_k \times a_{k+1} \times \cdots \times a_n$$

can be filled with bricks $a_1 \times \cdots \times a_n$. We show that this box is not a multiple of the brick. Let j be the smallest integer with $a_j = a_{k-1}$. Then no one of the numbers $a_1, \dots, a_{j-1}, a_{k-1} + a_k$ is divisible by any of the numbers $a_j, \dots, a_{k-1}, a_k, \dots, a_n$. (Note that $a_j = \cdots = a_{k-1}$, that $a_{k-1} + a_k$ is not divisible by a_{k-1} or a_k , nor by any multiple of a_k . And a_{k+1}, \dots, a_n are multiples of a_k , according to the way k was chosen.)

If the box $A_1 \times \cdots \times A_n$ is a multiple of the brick $a_1 \times \cdots \times a_n$, there can be at most $j - 1$ i 's such that A_i is no multiple of any of the numbers a_j, \dots, a_n . Therefore, the box constructed above is not a multiple of the brick. This completes the proof.

The material of Theorems 1 and 2 was presented in the form of problems in Hungarian in *Matematikai Lapok* 12 (1961) pp. 110–112, problem 109 and 13 (1962) pp. 314–317, problem 119. Problem 109, dealing with bricks $1 \times 2 \times 4$, was solved by G. Hajós, G. Katona, D. Szasz, I. Thiry, and the proposer. Problem 119 dealt with the n -dimensional case, and was solved by G. Hajós, G. Katona, D. Szasz, and the proposer.

The question of the bricks $1 \times 2 \times 4$ arose from a remark by the proposer's son F. W. de Bruijn who discovered, at the age of 7, that he was unable to fill his $6 \times 6 \times 6$ box by bricks $1 \times 2 \times 4$.