How to Detect a Counterfeit Coin

G. SHESTOPAL

Suppose that there is a pile of coins on a table, which are completely identical in appearance, but you are told that one of the coins is counterfeit. It differs from the others in weight, but you are not told whether it is lighter or heavier. You have at your disposal balance scales but no set of measuring weights. How can you identify the bad coin and determine its type (that is, whether it is lighter or heavier) in the minimal number of weighings?

Many of you have surely solved this problem for twelve coins. Figure 1 presents one possible solution. Corresponding to the three possible outcomes of the first weighing are three variants of the choice of coins for the second weighing: in the figure the left arrow corresponds to the case when the left pan is heavier, the middle arrow to the case when the pans balance, and the right arrow to the case when the right pan is heavier. Similarly, there are nine variants of the choice of coins for the third weighing. (The coins are numbered in the figure, and the letters "L" and "H" stand for "lighter" and "heavier", respectively.)

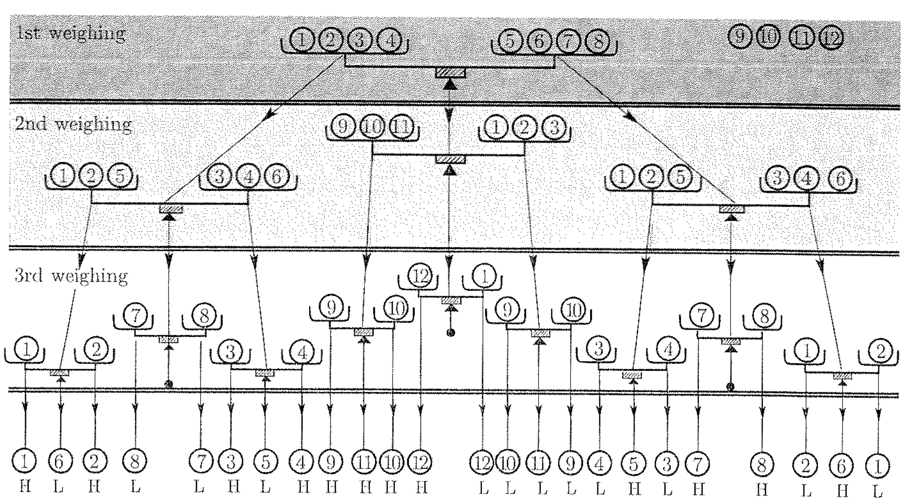

FIGURE 1

The Russian original is published in *Kvant* 1979, no. 10, pp. 21–25.

A characteristic feature of this solution is the dependence of the choice of coins for the next weighing on the result of the previous weighing.

Let us now pose the problem in general form.

We have $m \geq 3$ coins identical in appearance. All the coins except one have the same weight, and that one differs from the others in weight, though it is not known in which direction. What is the smallest number of weighings needed to find this coin and identify its type using balance scales without measuring weights?[1]

In the 1940s this problem attracted the attention of many mathematicians, mainly in England and the USA. In 1945 a solution of the problem appeared in the English journal *The Mathematical Gazette*, which is similar in purpose to *Kvant*. Its author, R. L. Goodstein, later became a well-known specialist in mathematical logic.

Goodstein indicated a method for determining the false coin and its type in $n \geq 3$ weighings if the number of coins satisfies

$$m \leq \tfrac{1}{2}(3^n - 2n + 3)$$

(note that $m \leq 12$ for $n = 3$).

However, it turned out that for $n > 3$ his solution is not the best possible: in n weighings one can detect the counterfeit coin and determine its type from a greater number of coins, namely,

$$m \leq \tfrac{1}{2}(3^n - 3).$$

This was observed independently by several mathematicians at once, and in the following year of 1946 the same journal published a fairly long list of their names and various degrees of success in the field of detecting false coins. In the same issue a best possible solution was presented by Freeman Dyson, who would later become famous as a theoretical physicist.

Dyson's idea was based on using the ternary number system: all the coins are marked by specially chosen ternary numbers, the markers enabling us to conveniently reflect the course of the successive weighings. An especially fascinating feature of this method of solution is the fact that the choice of coins for the next weighing is independent of the results of the previous weighings.

In subsequent years other solutions of the problem were published,[2] and Dyson's method was undeservedly forgotten. Therefore, it may be of interest to discuss it in detail.

Dyson's solution can be broken up into two steps.

A. If

$$m = \tfrac{1}{2}(3^n - 3), \quad n = 2, 3, \ldots,$$

then n weighings suffice to detect one false coin in a total number of m coins and to determine its type.

B. If the number of coins is

$$m < \tfrac{1}{2}(3^n - 3), \quad m \geq 3,$$

then n weighings suffice for the same goal.

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[1]Obviously, it does not make sense to solve the problem for $m < 3$: for $m = 1$ the coin is counterfeit, but the type is not known; for $m = 2$ the coins have different weights, and it is not possible to determine which one is counterfeit.

[2]For example, in the book *Probability and information* by A. Yaglom and I. Yaglom (3rd ed., "Nauka", Moscow, 1973; German transl. of 4th ed., VEB Deutscher Verlag Wiss., Berlin, 1984).

We treat each step separately.

First step

Suppose that the number of coins is $m = \frac{1}{2}(3^n - 3)$.

Let us consider all n-digit "ternary numbers" (n-words made up of the digits $0, 1, 2$): $00\ldots00, 00\ldots01, \ldots, 22\ldots22$. There are 3^n of them. We use them to mark the coins as described below. All these numbers are used as markers except the three consisting of identical digits, namely, $0\ldots0, 1\ldots1, 2\ldots2$.

We separate the totality of markers into pairs, a single pair being two complementary markers: the digits of the same order add up to 2 (that is, the sum of the two markers in a pair is $22\ldots22$).

A marker is called a *right* marker if the first pair of unequal digits starting from the left is 01, 12, or 20; otherwise, it is called a *left* marker. It is clear that in each pair of complementary markers one is always a right marker and the other a left marker.

We note that the number of pairs of markers is exactly equal to the number m of coins. Let us number the coins from 1 to m and arbitrarily assign to each coin one pair of markers. For example, twelve coins can be "marked" as indicated in Table 1.

TABLE 1

coin number	left marker	right marker
1	211	011
2	100	122
3	022	200
4	212	010
5	101	121
6	020	202
7	210	012
8	102	120
9	021	201
10	221	001
11	110	112
12	002	220

Let $M(i, 0)$, $M(i, 1)$, and $M(i, 2)$ denote the set of coins for which the ith digit of the corresponding right marker is equal to 0, 1, and 2 respectively.

It is easy to see that the number of coins is the same in each of the sets $M(i, 0)$, $M(i, 1)$, and $M(i, 2)$ and that these sets do not have common elements (see the table, for example).

The method devised by Dyson for weighing coins is as follows.

There are n weighings. For the ith weighing ($i = 1, \ldots, n$) all the coins in the set $M(i, 0)$ are placed on the left pan of the scales, and all the coins in the set $M(i, 2)$ on the right pan. The result of each weighing will be denoted by the digit 0 if the left pan is heavier, 1 if the pans balance, and 2 if the right pan is heavier

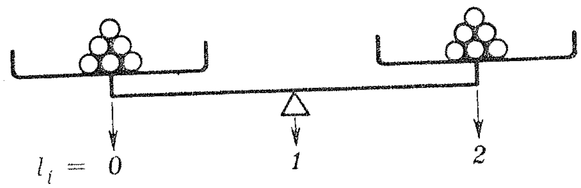

FIGURE 2

(Figure 2). The result of the ith weighing is denoted by l_i. From the digits $l_1, \ldots, l_n$ we form the marker[3] $l = l_1 \ldots l_n$.

It turns out that l is the marker of the false coin F, and that F is heavier than the others if l is a right marker and lighter if l is a left marker.

Indeed, let us see what happens when we carry out the ith weighing. As a result of this weighing the pans either balance or one is heavier.

If they balance, then the false coin is not on them, and hence it is in the set $M(i, 1)$. But this means that the ith digit of its right (and left) marker is 1, and the result $l_i = 1$ of the weighing indicates this.

If one of the pans is heavier, then the false coin is on the scales. Suppose, for example, that the right pan is heavier, that is, $l_i = 2$. This is possible in two cases:

- the false coin is on the right pan (and then it is heavier than the others), hence it is in the set $M(i, 2)$ and the ith digit of its right marker is 2, so that the result of the weighing coincides with the ith digit of its right marker;
- the false coin is on the left pan (and then it is lighter than the others), hence it is in the set $M(i, 0)$ and the ith digit of its right marker is 0, so that the result of the weighing coincides with the ith digit of its left marker.

The "symmetric" case when the left pan is heavier ($l_i = 0$) is completely analogous. Therefore, the marker $l = l_1 \ldots l_n$ formed as a result of the successive weighings is indeed the marker of the false coin, and is a right marker when that coin is heavier and a left marker when it is lighter, which is what was to be proved.

It is interesting to note that, as a rule, the type of the coin is determined before all the weighings have been performed: as soon as two distinct digits appear in the process of formation of the marker l.

As noted earlier, an essential feature of this method is that the choice of coins for each weighing does not depend on the results of the previous weighings.

For example, for twelve coins marked as in the table it is necessary to carry out the following three weighings: $(1, 4, 7, 10)$, $(3, 6, 9, 12)$; $(3, 6, 9, 10)$, $(2, 5, 8, 12)$; $(3, 4, 8, 12)$, $(2, 6, 7, 11)$.

Second step

We briefly sketch Dyson's method for the case $m < \frac{1}{2}(3^n - 3)$. If the markers are assigned arbitrarily to the coins in this case, then the sets $M(i, 0)$ and $M(i, 2)$ may have different number of coins. We therefore proceed as follows. Let us separate all the markers into groups of six each: in a single group we put the right markers obtained from each other by the cyclic permutation of digits $0 \to 1$, $1 \to 2$, $2 \to 0$, together with the left markers corresponding to them.

[3] Prove that this is really a marker, that is, that the digits $l_1, \ldots, l_n$ cannot all be the same.

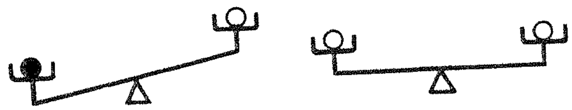

FIGURE 3

In each group there are three right markers and three left markers. We distinguish the group containing the right markers $00\ldots01, 11\ldots12, 22\ldots20$. Let us divide the coins into groups of three as long as possible, and mark them as follows. To the coins in a single group we assign the pairs of markers in a single group, and for the coins left over, if there are any, we use pairs of markers in the distinguished group. If one coin was left over and not put in a triple, then we assign it the right marker $11\ldots12$, and if two were left over, then we assign them the right markers $00\ldots01$ and $22\ldots20$ (and the corresponding left markers).

For such a marking of the coins the first $n-1$ weighings are carried out according to the old rules. How to do the last weighing you should figure out independently.

Dyson's method has been described. Let us now see that it is best possible in a certain sense. Namely, we show that if in n weighings it is possible to find the false coin among m coins and to determine its type, then $2m \leq 3^n - 3$. Of course, we shall not take into account the possibility of "luck": with luck, two weighings could be enough to find the false coin in any number of coins.

Accordingly, suppose that n weighings suffice.

We number the coins by $1, \ldots, m$ and take $2m$ slips of paper, on which we write all possible cases: the first coin is lighter, the first coin is heavier, the second coin is lighter, and so on. It is clear that this will use all $2m$ slips of paper, and no possible case will have been left out. The results of the weighings are denoted by the digits $0, 1, 2$ as specified above. Each weighing shows that some of the possible answers cannot be true but that some may still be true. Let coins be selected for the first weighing. Without actually performing this weighing, we write on each slip of paper the result of the first weighing that leaves its answer a possibility. Clearly, on each slip of paper one of the digits 0, 1, or 2 has been written. Thus, all the slips are divided into three groups, and so at least $2m/3$ slips are in the largest group. Therefore, no matter how we organize the first weighing, it can happen that afterwards at least $2m/3$ of the slips of paper are still possibilities.

Similarly, the second weighing sorts this group of possible answers into three subgroups, and thus at least $2m/9$ slips of paper are in the largest subgroup. In exactly the same way we can get $2m/3^n$ possible slips of paper in one group after n weighings. Consequently, if $2m/3^n > 1$, then in general the false coin and its type may not be determined in n weighings. Therefore, if n weighings suffice, then $2m \leq 3^n$.

But this is not yet all! We have not yet analyzed the case $2m = 3^n - 1$ (the even number $2m$ cannot be equal to $3^n - 2$ nor 3^n). The above arguments do not suffice for this case. Let us look more carefully at the first weighing. The number of papers on which 0 is written is clearly equal to the number on which 2 is written. If there are p pieces of paper for each, then 1 is written on $2m - 2p$ pieces of paper.

We remark that p is an even number. Indeed, if there are k coins on each of the left and right pans, then $p = 2k$. If p or $2m - 2p$ is greater than 3^{n-1}, then by the above considerations the remaining $n-1$ weighings may not suffice for determining the necessary slip of paper. But if $p = 2k \leq 3^{n-1}$ and $2m - 2p \leq 3^{n-1}$, then since 3^{n-1} is an odd number, we have $p \leq 3^{n-1} - 1$ and $2m - 2p \leq 3^{n-1} - 1$. But then $2m = (2m - 2p) + 2p \leq 3^{n-1} - 1 + 2(3^{n-1} - 1) = 3^n - 3$. Thus, if n weighings suffice, then $2m \leq 3^n - 3$.

We conclude with some problems that can be solved by Dyson's method.

PROBLEMS.

1. There are two groups of m_1 and m_2 coins, it is known that one coin is false, and it is lighter if it is in the first group and heavier if it is in the second group. What is the minimum number of weighings needed to find the false coin?

2. Suppose that in addition to the groups of m_1 and m_2 coins there is a group of m_3 standard (that is, definitely not false) coins. How many weighings are needed in this case?

3. We have m coins among which is a false coin (it is not known whether it is lighter or heavier), and we have one standard coin. Is it possible to determine the false coin and its type in n weighings if $m = \frac{1}{2}(3^n - 1)$? How many weighings are needed for other m?

4. We are given m coins and it is known that there is at most one false coin among them. How many weighings are needed to find the false coin and determine its type or to determine that there is no false coin?

5. In the case $2m < 3^n - 3$ the last weighing depended in general on the results of the previous weighings. Try to make it depend on the least possible number of weighings.