Math Seminar for Teachers
Jenya Soprunova, KSU

Stable Marriages

Problem 1. Given n boys and n girls, how many pairings are there?

Problem 2. You wish to assign four girls to two rooms, two per room. Come up with preference lists for the girls, where each of them ranks the remaining three as possible roommates (no ties allowed), such that no stable pairing exists.

Problem 3. Let A, B be two girls and α, β two boys. Come up with preference lists such that both possible pairings are stable.

Problem 4. Assume that among n boys and n girls, there is a boy and a girl who rank each other first. Show that there is only one feasible marriage for each of them, the one where they are together.

Problem 5. Show that Gale-Shapley algorithm for pairing up n boys and n girls takes at most $n^2 - n + 1$ rounds. (Hint: At each round at least one boy goes down his list. What happens at first round?)

Problem 6. Come up with a 3×3 ranking matrix for which Gale-Shapley algorithm takes 5 rounds. Explain why this is the largest number of rounds for $n = 3$.

Problem 7. For the following ranking matrix

	A	B	C	D
α	1,4	2,3	3,1	4,2
β	4,1	2,1	3,3	1,3
γ	1,3	2,4	3,4	4,1
δ	4,2	3,2	2,2	1,4

find all the feasible partners for α and then determine $\alpha's$ optimal and pessimal partners. (Do not run Gale-Shapley algorithm here.)

Problem 8. Run the Gale-Shapley algorithm for the ranking matrix below twice. First time let the boys $(\alpha, \beta, \gamma, \delta)$ propose, and then let the girls (A, B, C, D) propose.

	A	B	C	D
α	1,3	2,3	3,2	4,3
β	1, 4	4,1	3,3	2,2
γ	2,2	1,4	3,4	4,1
δ	4,1	2,2	3,1	1,4

Conclude that there is only one stable pairing for this ranking matrix. (Use the theorem that Gale-Shapley algorithm where boys propose is male-optimal and female-pessimal.)

Problem 9. For the following ranking matrix

$$\begin{bmatrix} 1,n & 2,n-1 & 3,n-2 & \dots & n,1 \\ n,1 & 1,n & 2,n-1 & \dots & n-1,2 \\ n-1,2 & n,1 & 1,n & \dots & n-2,3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2,n-1 & 3,n-2 & 4,n-3 & \dots & 1,n \end{bmatrix}$$

consider the pairing where each girl gets her kth choice for some fixed k with $1 \leq k \leq n$. Show that such a paring is stable. Hint: consider two pairs (A, α) and (B, β) and show that A and β would not run away. It is useful to notice that for each entry i, j in the ranking matrix we have $i + j = n + 1$.