relatively simple problems (but requiring ingenuity for solution) to problems whose solutions were a major achievement in combinatorics. Some of these problems will be considered in more detail in subsequent chapters.

1.1 Example: Perfect Covers of Chessboards

Consider an ordinary chessboard which is divided into 64 squares in 8 rows and 8 columns. Suppose there is available a supply of identically shaped dominoes, pieces which cover exactly two adjacent squares of the chessboard. Is it possible to arrange 32 dominoes on the chessboard so that no 2 dominoes overlap, every domino covers 2 squares, and all the squares of the chessboard are covered? We call such an arrangement a *perfect cover* of the chessboard by dominoes. This is an easy arrangement problem, and one quickly can construct many different perfect covers. It is difficult but nonetheless possible to count the number of different perfect covers. This number was found by Fischer[1] in 1961 to be $12,988,816 = 2^4 \times (901)^2$. The ordinary chessboard can be replaced by a more general chessboard divided into mn squares lying in m rows and n columns. A perfect cover need not exist now. Indeed, there is no perfect cover for the 3-by-3 board. For which values of m and n does the m-by-n chessboard have a perfect cover? It is not difficult to see that an m-by-n chessboard will have a perfect cover if and only if at least one of m and n is even or, equivalently, if and only if the number of squares of the chessboard is even. Fischer has derived general formulae involving trigonometric functions for the number of different perfect covers for the m-by-n chessboard. This problem is equivalent to a famous problem in molecular physics known as the *dimer problem*. It originated in the investigation of the absorption of diatomic atoms (dimers) on surfaces. The squares of the chessboard correspond to molecules, while the dominoes correspond to the dimers.

Consider once again the 8-by-8 chessboard and, with a pair of scissors, cut out two diagonally opposite corner squares, leaving a total of 62 squares. Is it possible to arrange 31 dominoes to obtain a perfect cover of this "pruned" board? Although the pruned board is very close to being the 8-by-8 chessboard, which has over twelve million perfect covers, it has no perfect cover. The proof of this is an example of simple but clever combinatorial reasoning. In an ordinary 8-by-8 chessboard the squares are alternately colored black and white, with 32 of the squares white and 32 of the squares black. If we cut out two diagonally opposite corner squares, we have removed two squares of the same color, say white. This leaves 32 black and 30 white squares. But each domino covers one black and one white square, so that 31 nonoverlapping dominoes on the board cover 31 black and 31 white squares. Therefore the pruned board has no perfect cover, and the reasoning above can be summarized by

$$31\,\boxed{\mathrm{B}\,|\,\mathrm{W}} \neq 32\,\boxed{\mathrm{B}} + 30\,\boxed{\mathrm{W}}\,.$$

More generally, one can take an m-by-n chessboard whose squares are alternately colored black and white and arbitrarily cut out some squares, leaving a pruned board. When does a pruned board have a perfect cover? For a perfect cover to exist the pruned board must have an equal number of black and white squares. But this is not sufficient, as the example in Figure 1.1 indicates.

W	×	W	B	W
×	W	B	×	B
W	B	×	B	W
B	W	B	W	B

Figure 1.1

Thus, we ask: What are necessary and sufficient conditions for a pruned board to have a perfect cover? We will return to this problem in Chapter 9 and will obtain a complete solution by applying the theory of matchings in bipartite graphs. There, a practical formulation of this problem is given in terms of assigning applicants to jobs for which they qualify.

There is another way to generalize the problem of a perfect cover of an m-by-n board by dominoes. Let b be a positive integer. In place of dominoes we consider 1-by-b pieces that consist of b 1-by-1 squares joined side by side consecutively. We call these pieces *b-ominoes*. Thus, a b-omino can cover b consecutive squares in a row or b consecutive squares in a column. In Figure 1.2, a 5-omino is illustrated. A 2-omino is simply a domino. A 1-omino is called a *monomino*.

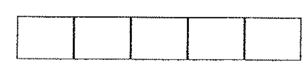

Figure 1.2. A 5-omino

[1]M.E. Fischer: Statistical Mechanics of Dimers on a Plane Lattice, *Physical Review*, 124 (1961), 1664-1672.

A *perfect cover* of an m-by-n board by b-ominoes is an arrangement of b-ominoes on the board so that (i) no two b-ominoes overlap, (ii) every b-omino covers b squares of the board, and (iii) all the squares of the board are covered. *When does an m-by-n board have a perfect cover by b-ominoes?* Since each square of the board is covered by exactly one b-omino, in order for there to be a perfect cover b must be a factor of mn. Surely, a sufficient condition for the existence of a perfect cover is that b be a factor of m or b be a factor of n. For if b is a factor of m, we may perfectly cover the m-by-n board by arranging m/b b-ominoes in each of the n columns, while if b is a factor of n we may perfectly cover the board by arranging n/b b-ominoes in each of the m rows. Is this sufficient condition also necessary for there to be a perfect cover? Suppose for the moment that b is a prime number and that there is a perfect cover of the m-by-n board by b-ominoes. Then b is a factor of mn and, by a fundamental property of prime numbers, b is a factor of m or b is a factor of n. We conclude that, at least for the case of a prime number b, an m-by-n board can be perfectly covered by b-ominoes if and only if b is a factor of m or b is a factor of n.

In case b is not a prime number, we have to argue differently. So suppose we have the m-by-n board perfectly covered with b-ominoes. We want to show that either m or n has a remainder of 0 when divided by b. We divide m and n by b obtaining quotients p and q and remainders r and s, respectively:

$$\begin{aligned} m &= pb + r, \quad \text{where} \quad 0 \le r \le b-1, \\ n &= qb + s, \quad \text{where} \quad 0 \le s \le b-1. \end{aligned}$$

If $r = 0$, then b is a factor of m. If $s = 0$, then b is a factor of n. By interchanging the two dimensions of the board, if necessary, we may assume that $r \le s$. We then want to show that $r = 0$.

1	2	3	$\cdots$	$b-1$	b
b	1	2	$\cdots$	$b-2$	$b-1$
$b-1$	b	1	$\cdots$	$b-3$	$b-2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
2	3	4	$\cdots$	b	1

Figure 1.3. Coloring of a b-by-b board with b colors

We now generalize the alternate black-white coloring used in the case of dominoes ($b = 2$) to b colors. We choose b colors which we label as 1, 2, ... , b. We color a b-by-b board in the manner indicated in Figure 1.3, and we extend this coloring to an m-by-n board in the manner illustrated in Figure 1.4 for the case $m = 10$, $n = 11$, and $b = 4$.

Each b-omino of the perfect covering covers one square of each of the b colors. It follows that there must be the same number of squares of each color on the board. We consider the board to be divided into three parts: the upper pb-by-n part, the lower left r-by-qb part, and the lower right r-by-s part. (For the 10-by-11 board in Figure 1.4, we would have the upper 8-by-11 part, the 2-by-8 part in the lower left, and the 2-by-3 part in the lower right.) In the upper part each color occurs p times in each column and hence pn times altogether. In the lower left part each color occurs q times in each row and hence rq times altogether. Since each color occurs the same number of times on the whole board, it now follows that each color occurs the same number of times in the lower right r-by-s part.

1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2

Figure 1.4. Coloring of a 10-by-11 board with four colors

How many times does color 1 (and, hence, each color) occur in the r-by-s part? Since $r \le s$, the nature of the coloring is such that color 1 occurs once in each row of the r-by-s part and hence r times in the r-by-s part. Let us now count the number of squares in the r-by-s part. On the one hand there are rs squares; on the other hand, there are r squares of each of the b colors and so rb squares altogether. Equating we get $rs = rb$. If $r \neq 0$, we cancel to get $s = b$, contradicting $s \le b-1$. So $r = 0$, as desired. We summarize as follows:

An m-by-n board has a perfect cover by b-ominoes if and only if b is a factor of m or b is a factor of n.

A striking reformulation of the preceding statement is the following: Call a perfect cover *trivial* if all the *b*-ominoes are horizontal or all the *b*-ominoes are vertical. Then *an m-by-n board has a perfect cover by b-ominoes if and only if it has a trivial perfect cover.* Note that this does not mean that the only perfect covers are the trivial ones. It does mean that if a perfect cover is possible, then a trivial perfect cover is also possible.

1.2 Example: Cutting a Cube

Consider a block of wood in the shape of a cube, 3 feet on an edge. It is desired to cut the cube into 27 smaller cubes, 1 foot on an edge. What is the smallest number of cuts in which this can be accomplished? One way of cutting the cube is to make a series of 6 cuts, 2 in each direction, while keeping the cube in one block as shown in Figure 1.5. But is it possible to use fewer cuts if the pieces can be rearranged between cuts? An example is also given in Figure 1.5 where the second cut now cuts through more wood than it would have if we had not rearranged the pieces after the first cut. Since the number of pieces, and thus the number of rearrangements, increases with each cut, this might appear to be a difficult problem to analyze.

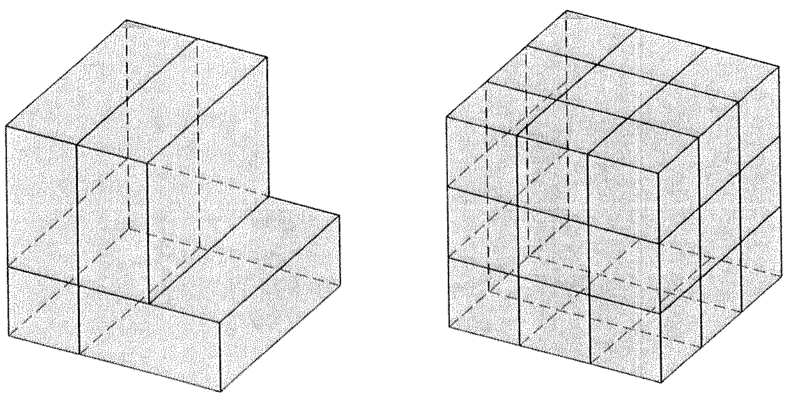

Figure 1.5

But let us look at it another way. Every one of the 27 small cubes except the one in the middle has at least one face that was originally part of one of the faces of the large cube. The cube in the middle has every one of its faces formed by cuts. Since it has 6 faces, 6 cuts are necessary to form it. Thus, at least 6 cuts are always necessary, and rearranging between cuts does not help. An energetic student might wish to investigate the number of different ways in which the cube can be cut into 27 smaller cubes, using only 6 cuts.

Another example, which combines features of Example 1.1 and the cube-cutting example, is the following: Consider a 4-by-4 chessboard that is perfectly covered with 8 dominoes. *Show that it is always possible to cut the board into two nonempty horizontal pieces or two nonempty vertical pieces without cutting through one of the 8 dominoes.* The horizontal or vertical line of such a cut is called a *fault-line* of the perfect cover. Suppose there is a perfect cover of a 4-by-4 board such that none of the three horizontal lines and three vertical lines that cut the board into two nonempty pieces is a fault-line. Let x_1, x_2, and x_3 be, respectively, the number of dominoes that are cut by the horizontal lines (see Fig. 1.6).

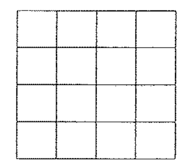

Figure 1.6

Because there is no fault-line, each of x_1, x_2, and x_3 is positive. A horizontal domino covers two squares in a row while a vertical domino covers one square in each of two rows. From these facts we conclude successively that x_1 is even, x_2 is even, and x_3 is even. Hence,

$$x_1 + x_2 + x_3 \geq 2 + 2 + 2 = 6,$$

and there are at least 6 vertical dominoes in the perfect cover. In a similar way, one concludes that there are at least 6 horizontal dominoes. Since $12 > 8$, we have a contradiction. Thus, it is impossible to perfectly cover a 4-by-4 board with dominoes without creating a fault-line.

1.3 Example: Magic Squares

Among the oldest and most popular forms of mathematical recreations are *magic squares.* A magic square of order n is an n-by-n array constructed out of the integers $1, 2, 3, \ldots, n^2$ in such a way that the sum of the integers in each row, in each column, and in each of the