WYTHOFF'S NIM AND FIBONACCI REPRESENTATIONS

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Our aim in what follows is to show how Fibonacci representations play a role in determining winning moves for Wythoff's Nim [1, 2] very analogous to the role of binary representations in Bouton's Nim [3]. The particulars of these two games can be found in the preceding references, but for the convenience of the reader, we briefly recount the rules of play.

In Bouton's Nim (usually referred to simply as nim) two players alternate picking up from a given collection of piles of counters (such as stones or coins). In his turn, a player must pick up at least one counter, so is never allowed a "pass." All counters picked up in one turn must come from a single pile, although the selection of pile can be changed from turn to turn. The number of counters picked up is constrained above by the size of the selected pile, but is otherwise an open choice on each move. The player who makes the last move (picking up the last counter) is declared the winner.

In Wythoff's Nim, only two piles are involved. On each turn a player may move as in Bouton's Nim, but has the added option of picking up from both piles, provided he picks up an equal number of counters from the two piles. As in Bouton's Nim, the winner is the player who makes the last move.

As has been known since Wythoff's original paper [1], the strategy for Wythoff's Nim consists of always leaving the opponent one of a sequence of pairs

$$(1,2),\ (3,5),\ (4,7),\ \cdots$$

which are defined inductively. Generating formulas have been found for these pairs, but they involve computations with irrational quantities.

Thus a certain inequity is seen to exist with regard to the computation of play for the two games. In Bouton's Nim it is possible to determine correct play solely on the basis of the cardinalities of the constituent piles by way of their binary representations; no other information is necessary. In contrast, the traditional play of Wythoff's Nim requires having or computing the table of "safe" pairs to an appropriately advanced position. An analogous approach to playing Bouton's Nim would require the inductive generation of safe configurations up to an appropriately advanced stage, a process which, although altogether well-defined and straightforward, is quite complex in that case.

Recent researches into Fibonacci representations [4, 5, 6, 7 and more] have coincidentally turned up the safe pairs for Wythoff's Nim as being quite fundamental to the analysis of the Fibonacci number system. An immediate and explicit by-product of these investigations is a method of computing moves for Wythoff's Nim using Fibonacci representations. Ostensibly, this method places the two games on an equal footing with respect to the computation of play; in fact, there is a residual disparity, related to the Fibonacci representations. At the present time, the determination of Fibonacci representations requires a listing or calculation of Fibonacci numbers to an appropriately advanced stage, a calculation which resembles the very aspect of the present method which we might hope to eliminate. In the case of binary representations, a succession of divisions provides the needed representations, eliminating the necessity of computing tables of powers of 2. The existence and/or determination of an analogous algorithm for the determination of Fibonacci representations is therefore a related question of interest in what follows.

As has been stated, the method we are about to describe is an explicit consequence of the material in [4] and [5]. Our contribution consists of making available a description and proof of the method which is self-contained, and on the same level as Bouton's treatment [1]. This will remove the necessity for a good deal of background which might otherwise exclude this information from many to whom this same information will be comprehensible in its present form.

We begin by reviewing some basic facts about the Fibonacci number system. A Fibonacci representation consists of a finite sequence of zeroes and ones, read positionally from right to left. A one in the i^{th} position denotes the presence of the i^{th} Fibonacci number F_i, where we adopt the convention $F_1 = 1$, $F_2 = 1$, $F_3 = 2$ and so forth. The number represented is determined by summing the Fibonacci numbers whose presence is indicated by a one. Thus 100000 denotes $F_6 = 8$, 11000 denotes $F_5 + F_4 = 8$ and 10101 denotes $F_5 + F_3 + F_1 = 8$. Clearly uniqueness is not generally assured.

A Fibonacci representation is said to be *canonical* if it satisfies two conditions: (i) the representation contains no adjacent ones and (ii) F_1 is not present (although F_2 may be) meaning that there is not a one in the first position. The canonical form turns out to exist and to be unique for each positive integer and is the representation resulting from the following algorithm: given n, determine the largest $F_k \leqslant n$ and place a 1 in position k. Repeat for the residue $n - F_k$ and continue to repeat, driving the residue to zero. (Be sure to denote the number 1 by F_2 instead of F_1, if it is needed.) Of course, the F_i not used correspond to zeroes in the canonical representation.

A Fibonacci representation is said to be *second canonical* if it satisfies condition (i) of the canonical representation and in addition (ii)' the right-most one of the representation is in an odd-numbered position. The second canonical form turns out to exist and to be unique for each positive integer and is the representation resulting from the following algorithm: given n, determine the canonical representation of $n - 1$ and then add 1 "in Fibonacci," meaning that any pair of adjacent ones is rounded up to a single one in the position immediately to the left of the pair. In the case of 8, 100000 is the canonical representation and 10101 is the second canonical representation.

The canonical and second canonical Fibonacci representations are lexicographic, which means the following. If m and n are positive integers, then $m < n$ iff the left-most position in which the canonical Fibonacci representations of m and n differ contains a one in the representation of n and a zero in that of m. The same is also true of the second canonical Fibonacci representations. What this amounts to is this: just as in the case of conventional bases, one can determine the larger of two numbers "at sight" by comparing their representations.

If the reader is willing to accept the assertions we have made concerning the canonical and second canonical Fibonacci representations, we can produce an otherwise complete account of the winning strategy for Wythoff's Nim.

We begin with some definitions. Let n be a positive integer which is represented canonically (not necessarily second canonically) in Fibonacci. We shall call n an *A-number* if the right-most one in the representation of n occurs in an even-numbered position; otherwise we call n a *B-number.* Obviously every positive integer is either an *A-number* or a *B-number,* but never both.

Let a and b be positive integers, $a < b$. The pair (a,b) will be called a *safe pair* if the following two conditions are satisfied: (i) the smaller number a is an A-number and (ii) the canonical Fibonacci representation of b is equal to that of a with a zero adjoined at the right end. For convenience we also agree to consider (0,0) a safe pair. Thus, for example, (12, 20) is a safe pair, since in canonical Fibonacci this is written (101010, 1010100). Notice that if (a,b) is a safe pair other than (0,0), b must be a B-number. A pair which is not safe is *unsafe.*

We shall prove the following two theorems.

Theorem 1. If (a,b) is a safe pair, then every pair (c,d) which is derived from (a,b) by a legal move is unsafe.

Theorem 2. If (c,d) is an unsafe pair, then there exists a safe pair (a,b) derivable from (c,d) by a legal move.

Since (0,0) is a safe pair, it is clear that the winning strategy for Wythoff's Nim consists of always leaving one's opponent a safe pair. Thus the first player who can establish a safe pair is the winner, provided he continues to play correctly.

We introduce one last bit of terminology for convenience. Given any Fibonacci representation, the *left shift* of that representation is the representation obtained by adjoining a zero at the right end. The *right shift* is the representation obtained by deleting the digit at the right end. We now prove some lemmas which will considerably expedite the proofs of the theorems.

Lemma 1. If (a,b) is any safe pair different from (0,0), the right shift of the canonical representation of a yields the second canonical representation of $b - a$.

Proof. Clearly the right shift of any A-number is in second canonical form. Thus we need only show that the right shift of a represents $b - a$. Each one in the representation of b is equivalent to a pair of ones in the two immediately adjacent positions on its right. (This is simply a way of saying that $F_n = F_{n-1} + F_{n-2}$ if $n \geq 3$, and since b is a B-number in canonical form, the position of each one is 3 or greater.)

Suppose we denote the right shift operation by $\rightarrow$. From the preceding remark, it is clear that

$$b = \vec{b} + \vec{\vec{b}} .$$

By the definition of a safe pair, $\vec{b}$ is the canonical representation of a, so this equation is the same as

$$b = a + \vec{a} ,$$

which is the desired conclusion.

Now let $\leftarrow$ denote the left shift.

Lemma 2. For each positive integer n, there is exactly one safe pair (a_n, b_n) with $b_n - a_n = n$. If n is represented in second canonical form, then $a_n = \overleftarrow{n}$ and $b_n = \overleftarrow{a_n} = \overleftarrow{\overleftarrow{n}}$, with a_n and b_n in canonical form.

Proof. Let (a,b) and (a',b') be safe pairs with $b - a = b' - a' = n$. By Lemma 1, if a and a' are represented canonically, both $\vec{a}$ and $\vec{a'}$ are the second canonical representation of n. But the latter is unique and thus the canonical representations of the A-numbers a and a' are identical. It follows that there can be no more than one safe pair (a_n, b_n) with $b - a = n$ for a given n.

Clearly the left shift of any second canonical representation is the canonical representation of an A-number. Therefore, given any positive integer n, set $a_n = \overleftarrow{n}$ and $b_n = \overleftarrow{\overleftarrow{n}}$ with n in second canonical form to obtain a safe pair (a_n, b_n) such that $b_n - a_n = n$.

Lemma 2 has the following corollary.

Corollary. If $m < n$ then $a_m < a_n$ and $b_m < b_n$.

Proof. Fibonacci representations are lexicographic.

Since no A-number can be a B-number, this corollary yields another.

Corollary. Each positive integer belongs to exactly one safe pair.

Proof of Theorem 1. Let (a_n, b_n) be a safe pair. By the rules of the game, a legal move must either reduce a_n or b_n alone or reduce both a_n and b_n by the same amount. If a_n alone is reduced, the resultant pair still contains b_n so cannot be safe by the preceding corollary; likewise for b_n. If both a_n and b_n are reduced to obtain a pair (a,b), then $b - a = b_n - a_n$, so that (a,b) cannot be safe because of Lemma 2.

Proof of Theorem 2. Suppose that (a,b) is an unsafe pair. If $a = b$, the pair can be reduced to the safe pair (0,0). If $a \neq b$ we assume $a < b$. Represent a and b canonically. If a is a B-number, reduce b to $\vec{a}$. If a is an A-number and $b > \overleftarrow{a}$, reduce b to $\overleftarrow{a}$. If a is an A-number and $b < \overleftarrow{a}$, then $b - a < \overleftarrow{a} - a$. Let $m = b - a > 0$ and $n = \overleftarrow{a} - a$, so $m < n$ and $(a, \overleftarrow{a}) = (a_n, b_n)$. By the corollary, $a_m < a_n = a$, so that a can be reduced to a_m. An equal reduction in b necessarily produces b_m, since by definition, (a_m, b_m) is the unique safe pair with $b_m - a_m = m$.

If we put together the proof of Theorem 2 and the statement of Lemma 2, we arrive at the following specific algorithm for playing Wythoff's Nim.

0. Given a pair (a,b), represent a and b canonically in Fibonacci. If (a,b) is a safe pair, concede (if you think your opponent knows what he is doing). Otherwise, proceed to 1.

1. If the smaller number of the pair is a B-number, reduce the larger to that quantity represented by the right shift of the canonical Fibonacci representation of the smaller.

2. If the smaller number of the pair is an A-number, and if in addition the larger number of the pair exceeds that quantity represented by the left shift of the canonical representation of the smaller, reduce it to the latter quantity.

3. If neither 0, 1 nor 2 holds, determine the second canonical Fibonacci representation of the positive difference of the members of the pair. A left shift on this representation will produce an A-number and another

left shift will produce a *B*-number, the two of which constitute a safe pair obtainable from (a,b) by the reduction of a and b by an equal amount.

We conclude by illustrating each of these cases. Suppose $a = 10$, $b = 15$. Then in canonical form, a = 100100 and b = 1000100. Thus a is a *B*-number (case 1) so we reduce b to $\vec{a}$ = 10010. The result is the safe pair (6,10).

Suppose $a = 9$, $b = 20$. Then a = 100010 and b = 1010100. Here a is an *A*-number with $\overleftarrow{a}$ = 1000100, which is less than b by inspection (case 2). We therefore reduce b to $\overleftarrow{a}$ obtaining the safe pair (9, 15).

Suppose $a = 24$, $b = 32$. In canonical form, a = 10001000 and b = 10101000. Here a is an *A*-number with $\overleftarrow{a}$ = 100010000 $> b$ (case 3). Hence we compute $b - a = 8$. The canonical representation of 7 is 10100 so the second canonical representation of 8 is 10101. This gives the canonical representation 101010 for a_8 and 1010100 for b_8, yielding the safe pair (12, 20), which is obtained by reducing both 24 and 32 by 12.

FINAL NOTE. We are indebted to Mr. Martin Gardner who furnished the additional reference [10] upon reading a preprint of this manuscript. References [11, 12, 13] are cited in [10]. The connection with Fibonacci representations we have discussed is given (in very definite form) in [12] and is generalized in [10]. However, in neither case is the connection with second canonical representations discussed, which is the key to actual computation of play. In this connection reference [13] is usable, but more complicated than the second canonical approach herein.

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