arXiv:math.AG/0401074 v1 8 Jan 2004

ZEROS OF SYSTEMS OF EXPONENTIAL SUMS AND TRIGONOMETRIC POLYNOMIALS

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Abstract. Gelfond and Khovanskii found a formula for the sum of the values of a Laurent polynomial at the zeros of a system of n Laurent polynomials in $(\mathbb{C}^\times)^n$ whose Newton polyhedra have generic mutual positions. An exponential change of variables gives a similar formula for exponential sums with rational frequencies. We conjecture that this formula holds for exponential sums with real frequencies. We give an integral formula which proves the existence-part of the conjectured formula not only in the complex situation but also in a very general real setting. We also prove the conjectured formula when it gives answer zero, which happens in most cases.

1. Motivation and Summary

Algebraic geometry is concerned with the study of zero sets of algebraic polynomials $f(x) = \sum_\alpha c_\alpha x^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, $x = (x_1, \dots, x_n)$. If we allow the exponents α to be real vectors, f becomes a multi-valued function. This can be remedied by an exponential change of variables $x_i = \exp 2\pi z_i$, after which we obtain a single-valued exponential sum of the form $\sum_\alpha c_\alpha \exp 2\pi\alpha z$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ is the vector of frequencies, $z = (z_1, \dots, z_n)$ is the vector of variables, and αz is the standard scalar product.

Some results from algebraic geometry can be generalized to this wider class of functions. For example, Bernstein's theorem states that the number of zeros of a generic system of n algebraic equations in $(\mathbb{C}^\times)^n$ with a fixed collection of Newton polyhedra is equal to $n!$ times the mixed volume of the Newton polyhedra of the system. A system of n exponential equations in n variables usually has infinitely many isolated zeros, thus one has to study the distribution of these zeros in order to obtain finite invariants. O. Gelfond proved that the mean number of complex zeros of a system of n exponential sums in n variables with real frequencies whose Newton polyhedra have sufficiently general mutual positions (so-called developed collection of Newton polyhedra), is equal to $n!$ times the mixed volume of the Newton polyhedra of the system [8].

O. Gelfond and A. Khovanskii found a formula for the sum of the values of a Laurent polynomial at the zeros of a system of n Laurent polynomials in $\mathbb{C}^n$ with

2000 *Mathematics Subject Classification.* 14P15, 33B10.

Key words and phrases. Exponential sums, trigonometric polynomials, quasiperiodic functions, mean value.

a developed collection of Newton polyhedra [5, 6]. This formula splits into two components. One of them is geometrical and reflects the mutual positions of the Newton polyhedra of the system, while the other component is expressed explicitly in terms of the coefficients of the polynomial and the system.

An exponential change of variables gives a similar formula for exponential sums with rational frequencies. We conjecture that this formula also holds for exponential sums with real frequencies. Here is some evidence for this. If the exponential sum that we are summing up is identically equal to one, the formula follows from combining two results: Gelfond's generalization of Bernstein's theorem [8] and the new formula for the mixed volume [11]. In [13] the conjectured formula is proved in dimension one.

The conjectured formula, first of all, implies that the mean value exists. We prove the existence not only in the complex situation but also in a very general real setting by providing an integral formula for the mean value.

If the frequencies of the exponential sum that we are summing up are not commensurate with the frequencies of the system the conjectured formula states that the mean value is equal to zero. We show that this is actually true and therefore prove the conjectured formula in most cases. For example, if the exponential sum that we are summing up is a single exponent $\exp 2\pi\alpha z,\ z \in \mathbb{C}^n,\ \alpha \in \mathbb{R}^n$, then the formula is proved for all values of α except for a countable set in $\mathbb{R}^n$.

Our arguments represent a combination of real analytic geometry and ergodic theory. They are based on two theorems from completely different parts of mathematics: the cell-decomposition theorem for subanalytic sets (Appendix B) and Weyl's equidistribution law for multidimensional trajectories in the real torus (Appendix A). Weyl's equidistribution law for one-dimensional trajectories is a classical theorem which was published in 1916 (see [16]). The proof of the corresponding law for multidimensional trajectories in [12] is a direct generalization of Weyl's original argument.

Remarks. Similar ideas first appeared in Weyl's papers [14] and [15] where he solves the mean motion problem. S. Gusein-Zade and A. Esterov were dealing with related questions and were using close techniques in [9, 10, 3] to prove the existence of the mean Euler characteristic and mean Betti numbers of level sets and sets of smaller values of a quasiperiodic function. Their motivation is completely different from ours: it comes from the analysis of some models of chaotic behavior appearing in quasicrystal structures. See also a recent paper [1] of V. I. Arnold for a close discussion.

Acknowledgements. This paper is a part of the author's Ph.D. thesis [12] defended in spring 2002. I would like to thank my thesis advisor Askold Khovanskii for stating the problem and for his constant attention to this work.

2. The Gelfond–Khovanskii formula and its conjectured generalization to the case of exponential sums with real frequencies

Let $\Delta_1, \dots, \Delta_n$ be convex polyhedra in $\mathbb{R}^n$ and $\Delta = \Delta_1 + \dots + \Delta_n$ be their Minkowski sum.

Definition 2.1. A collection of faces $\{\Gamma_j : \Gamma_j \subset \Delta_j, j = 1 \dots n\}$ is *coordinated* if there exists a nonzero linear function on $\mathbb{R}^n$ whose maximum on the polyhedron Δ_j is attained exactly on the face $\Gamma_j \subset \Delta_j$ for all $j = 1 \dots n$. A collection of polyhedra $\Delta_1, \dots, \Delta_n$ is *developed* if none of the polyhedra is a vertex, and in any coordinated collection of faces there is at least one vertex.

Fix a system

$$P_1(z) = \dots = P_n(z) = 0, \quad z \in \mathbb{C}^n$$

of n Laurent equations with a developed collection of Newton polyhedra $\Delta_1, \dots, \Delta_n$. The Gelfond–Khovanskii formula states that the sum of the values of a Laurent polynomial Q over the zeros of the system in $(\mathbb{C}^\times)^n$ is equal to

$$(-1)^n \sum_\alpha k_\alpha C_\alpha,$$

where the summation is performed over the vertices α of $\Delta = \Delta_1 + \dots + \Delta_n$, k_α is the combinatorial coefficient that corresponds to the vertex α (the combinatorial coefficients are integers that reflect the mutual position of the Newton polyhedra $\Delta_1, \dots, \Delta_n$ of the system, see [5, 6] for the definition), and C_α is an explicit Laurent polynomial in the coefficients of $P = P_1 \cdots P_n$ and Q. This result was announced in [5], and a proof was given in [6].

If we allow the exponents of the Laurent polynomials to be real vectors, we obtain multi-valued functions. After an exponential change of variables they become single-valued exponential sums with real frequencies. Now we make some preparations before we formulate the conjectured generalization of the Gelfond–Khovanskii formula to the case of exponential sums.

Let Λ be a finite set in $\mathbb{R}^n$. An *exponential sum with the spectrum* Λ is a function $F : \mathbb{C}^n \to \mathbb{C}$ of the form

$$F(z) = \sum_{\alpha \in \Lambda} c_\alpha \exp 2\pi \alpha z,$$

where the summation is performed over the frequencies $\alpha \in \Lambda$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, c_α are nonzero complex numbers, and αz is the standard scalar product. The *Newton polyhedron* of an exponential sum is the convex hull $\Delta(\Lambda)$ of its spectrum Λ.

Fix a system of exponential sums

$$F_1(z) = \dots = F_n(z) = 0, \quad z \in \mathbb{C}^n \tag{2.1}$$

with a developed collection of Newton polyhedra $\Delta_1, \dots, \Delta_n$ in $\mathbb{R}^n$. Due to the assumption that the collection of the polyhedra is developed, there exists $R > 0$ such that all the zeros of the system lie in a strip $S_R \times \operatorname{Im} \mathbb{C}^n$, where $S_R \subset \operatorname{Re} \mathbb{C}^n$ is

a ball of radius R centered at the origin. This implies that all the zeros are isolated (see Theorem 5.1).

Let G be another exponential sum with real frequencies. The sum of the values at the zeros of a system in the exponential case is replaced with the result of averaging G over the zeros of the system along the imaginary subspace. Let $\Omega \subset \operatorname{Im} \mathbb{C}^n$ be a bounded set with nonzero volume. For $\lambda \in \mathbb{R}$, define $S_\Omega(\lambda)$ to be the sum of the values of G at the zeros of (2.1) (counting multiplicities) that belong to the strip $\mathbb{R}^n \times \lambda\Omega \subset \mathbb{C}^n$.

Definition 2.2. The *mean value* M_Ω of G over the zeros of the system (2.1) is the limit of $S_\Omega(\lambda)/\operatorname{Vol}(\lambda\Omega)$, as λ approaches infinity.

The Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_n$ is the Newton polyhedron of the product $F = F_1 \cdots F_n$. The exponent $\exp 2\pi\alpha z$ that corresponds to a vertex α in Δ appears in F with a nonzero coefficient d_α. Let $\widetilde{F} = F/(d_\alpha \exp 2\pi\alpha z)$. The constant term of the exponential sum $\widetilde{F}$ is equal to one. We define the exponential series for $1/\widetilde{F}$ by the formula

$$1/\widetilde{F} = 1 + (1 - \widetilde{F}) + (1 - \widetilde{F})^2 + \cdots .$$

Since each exponent appears with a nonzero coefficient in a finite number of terms, the coefficients of this series are well-defined. Let C_α be the constant term in the formal product of this series and $(1/d_\alpha)\exp(-2\pi\alpha z) G \det(\frac{\partial F}{\partial z})$.

Conjecture 2.3. The mean value M_Ω of an exponential sum G over the zeros of the system (2.1) is equal to

$$\frac{1}{(-2\pi)^n} \sum_\alpha k_\alpha C_\alpha,$$

where the summation is performed over the vertices α of the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_n$, C_α are the constant terms of the series as defined above, and k_α are the combinatorial coefficients.

As we mentioned in the summary, this formula is proved in the following three cases:

1. The frequencies α are rational (the formula is obtained by an exponential change of variables from the algebraic case).
2. $G = 1$ (the formula follows from combining two results: Gelfond's generalization of Bernstein's theorem [8] and the new formula for mixed volume [11]).
3. $n = 1$ (see [13]).

In the exponential case even the existence of the mean value is not obvious. We give an integral formula for the mean value which proves the existence not only in the complex situation (Theorem 5.2) but also in a very general real setting (Theorem 4.3).

Assume that the frequencies of the exponential sum that we are summing up are not commensurate with the frequencies of the system. Then there is no constant term

in the series defined above, that is, all C_α are equal to zero, and the conjectured formula states that the mean value is equal to zero. Using the integral representation, we show in Corollary 5.3 that this is actually true. This proves the conjectured formula in most cases. For example, if the exponential sum that we are summing up is a single exponent $\exp 2\pi\alpha z$, then we proved the formula for all values of α except for a countable set in $\mathbb{R}^n$.

3. Averaging over the isolated intersections of a subanalytic set with a dense orbit on $\mathbb{T}^N$

Here we present a construction and formulate a theorem on which all our results are based. Let $\widetilde{V}$ be a subanalytic subset of the real torus $\mathbb{T}^N$ and $\mathcal{O}$ a dense multidimensional orbit on $\mathbb{T}^N$. We prove an integral formula for the mean value of a bounded subanalytic function on $\widetilde{V}$ over the isolated intersections of the set $\widetilde{V}$ and the orbit $\mathcal{O}$. This formula, in particular, implies that the mean value always exists.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. A linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ defines an action of $\mathbb{R}^n$ on $\mathbb{T}^N$ by

$$x : \varphi \mapsto (\varphi + \Phi(x)) \bmod \mathbb{Z}^N.$$

Let $\mathcal{O}$ be an orbit of this action. We require that the orbits are dense in the torus, which means that there are no integral vectors orthogonal to the n-plane $\Phi(\mathbb{R}^n) \subset \mathbb{R}^N$ (Appendix A). We will also assume that Φ is injective, that is, the orbits are n-dimensional.

Let $\widetilde{V}$ be a subanalytic subset of $\mathbb{T}^N$ and $\widetilde{T}$ a bounded subanalytic function on $\widetilde{V}$. Set $V := \Phi^{-1}(\widetilde{V} \cap \mathcal{O}) \subset \mathbb{R}^n$, and define a function $T(x) = \widetilde{T}(\Phi(x))$ on V. Let Ω be a bounded subset of $\mathbb{R}^n$ with nonzero volume. For $\lambda > 0$, define $S_\Omega(\lambda)$ to be the sum of the values of T at the isolated points of V that belong to $\lambda\Omega$.

Definition 3.1. The *mean value* M_Ω of $\widetilde{T}$ over the isolated intersections of $\widetilde{V}$ and $\mathcal{O}$ (in the topology of $\mathcal{O}$) is the limit:

$$M_\Omega = \lim_{\lambda\to\infty} \frac{S_\Omega(\lambda)}{\mathrm{Vol}(\lambda\Omega)}.$$

Let $p : \mathbb{R}^N \to L$ be the orthogonal projection to the linear subspace $L \subset \mathbb{R}^N$ orthogonal to $\Phi(\mathbb{R}^n)$. Define $M_{N-n}(\widetilde{V})$ to be the smooth $(N-n)$-dimensional part of $\widetilde{V}$, transversal to the orbit. Here is the precise three step definition (see Figure 1 below where we assume that the direction of the orbit is vertical):

(1) Let $\widetilde{V}_0$ be the set of all points $\varphi \in \widetilde{V}$ such that the intersection of $\widetilde{V}$ with the n-plane through φ parallel to the orbit is locally just the point φ itself.
(2) Let $\widetilde{V}_1$ be the set of all points $\varphi \in \widetilde{V}_0$ such that $\widetilde{V}_0$ is a C^1-manifold of dimension $N-n$ in a small neighborhood of φ.
(3) Let $M_{N-n}(\widetilde{V})$ be the set of all points $\varphi \in \widetilde{V}_1$ such that the projection p is regular at each of these points.

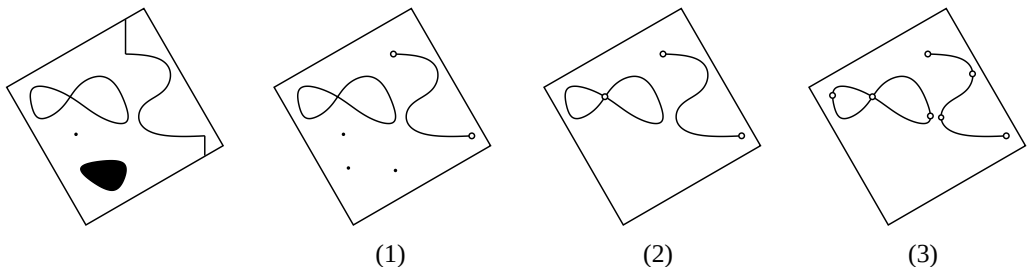

FIGURE 1. Definition of $M_{N-n}(\widetilde{V})$

Let $A^j = \Phi(e_j)$, where e_j is the j-th vector in the standard basis for $\mathbb{R}^n$. The quotient of the standard volume form on $\mathbb{R}^N$ by the image under Φ of the standard volume form on $\mathbb{R}^n$ defines a volume form $\omega = d\varphi(\dots, A^1, \dots, A^n)$ and an orientation on L. Then $M_{N-n}(\widetilde{V})$ is a C^1-manifold, and the projection p defines an $(N-n)$-form $p^*\omega$ on $M_{N-n}(\widetilde{V})$.

Definition 3.2. The *$\mathcal{O}$-transversal $\widetilde{T}$-weighted volume* of $\widetilde{V}$ is the integral of the form $\widetilde{T}p^*\omega$ over $M_{N-n}(\widetilde{V})$.

Definition 3.3. We say that the *fractal dimension* of a set X is less than or equal to n if there exist constants c and $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the set X can be covered by no more than $c\epsilon^{-n}$ balls of radius ϵ. We say that the fractal dimension is less than n if it is less than or equal to $n - \delta$ for some positive δ.

Our main result is the following theorem:

Theorem 3.4. *Let $\mathcal{O}$ be an n-dimensional dense orbit in the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$, $n < N$, $\widetilde{V}$ a subanalytic subset of $\mathbb{T}^N$, and $\widetilde{T}$ a bounded subanalytic function on $\widetilde{V}$. Let Ω be a bounded subset of $\mathbb{R}^n$ with nonzero volume whose boundary has fractal dimension less than n.*

Then the mean value M_Ω of $\widetilde{T}$ over the isolated intersections of $\widetilde{V}$ and $\mathcal{O}$ always exists and equals the $\mathcal{O}$-transversal $\widetilde{T}$-weighted volume of $\widetilde{V}$.

We prove this theorem in Section 6.

4. Zeros of systems of trigonometric polynomials

We apply Theorem 3.4 to prove that the mean value of a quasiperiodic trigonometric polynomial T over the isolated points of a quasiperiodic semitrigonometric set V (a set described by quasiperiodic trigonometric polynomials, see the precise definition below) always exists. We obtain an integral representation for the mean value from which we conclude that in the case when the frequencies of T are not commensurate with the frequencies of the quasiperiodic trigonometric polynomials that describe V, the mean value is equal to zero.

4.1. **Definitions.** Let $x \in \mathbb{R}^n$. A *quasiperiodic trigonometric polynomial* is a trigonometric polynomial of the form

$$T(x) = \sum_{k=1}^{p} c_k \cos 2\pi\alpha_k x + d_k \sin 2\pi\alpha_k x$$

with real coefficients $c_k, d_k \in \mathbb{R}$ and real frequencies $\alpha_k \in \mathbb{R}^n$. Here $\alpha_k x$ is the standard scalar product.

A subset of $\mathbb{R}^n$ is called quasiperiodic semitrigonometric if it is described by quasiperiodic trigonometric equations and inequalities. More precisely, we say that $V \subset \mathbb{R}^n$ is *quasiperiodic semitrigonometric* if it can be represented in the form:

$$V = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} V_{ij},$$

where each V_{ij} is either $\{x \in \mathbb{R}^n \,|\, T_{ij}(x) = 0\}$ or $\{x \in \mathbb{R}^n \,|\, T_{ij}(x) > 0\}$, for some quasiperiodic trigonometric polynomials T_{ij}.

Let T be a quasiperiodic trigonometric polynomial, V a quasiperiodic semitrigonometric set in $\mathbb{R}^n$ described by T_{ij}, and Ω a bounded subset of $\mathbb{R}^n$ with nonzero volume. For $\lambda > 0$, define $S_\Omega(\lambda)$ to be the sum of the values of T at the isolated points of V that belong to $\lambda\Omega$. The *mean value* M_Ω of T over the isolated points of V is defined by

$$M_\Omega = \lim_{\lambda\to\infty} \frac{S_\Omega(\lambda)}{\mathrm{Vol}(\lambda\Omega)}.$$

4.2. **Averaging a quasiperiodic trigonometric polynomial over the isolated points of a quasiperiodic semitrigonometric set.** Consider the set of all frequencies of T and T_{ij}. This set generates a subgroup $\mathcal{A}$ of $(\mathbb{R}^n, +)$, which is a direct sum of infinite cyclic subgroups:

$$\mathcal{A} = (A_1) \oplus \cdots \oplus (A_N).$$

That is, there exist $A_1, \ldots, A_N \in \mathbb{R}^n$ with no integral relations, such that each of the frequencies is an integral combination of $A_1, \ldots, A_N$.

Define a linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ by $\Phi(x) = (A_1 x, \ldots, A_N x)$. Then $\mathbb{R}^n$ acts on $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ by

$$x : \varphi \mapsto (\varphi + \Phi(x)) \bmod \mathbb{Z}^N.$$

Let $\mathcal{O}$ be the orbit of this action through the origin. Since each of the frequencies α_k is an integral combination of $A_1, \ldots, A_N$, the functions T and T_{ij} are the restrictions to $\mathcal{O}$ of some trigonometric polynomials $\widetilde{T}$ and $\widetilde{T}_{ij}$ defined on $\mathbb{T}^N$. For example, $\cos(2\pi\alpha x)$ is the restriction of $\cos 2\pi \sum m_i\varphi_i$, where $\alpha = \sum m_i A_i$ for $m_i \in \mathbb{Z}$.

To apply Theorem 3.4 here we need the orbit to be dense (Lemma 4.1), the mapping Φ to be injective (Lemma 4.2), and the dimension n of the orbit to be less than N (if Φ is injective then $n \le N$; $N = n$ corresponds to the periodic case which we treat in Remark 4.4).

Lemma 4.1. *There are no nonzero integral vectors orthogonal to the plane of the orbit $\mathcal{O}$. Therefore, $\mathcal{O}$ is dense in the torus $\mathbb{T}^N$.*

Proof. Let A be the $N \times n$ matrix whose rows are the generators $A_1, \dots, A_N$ of $\mathcal{A}$. We denote the columns of this matrix by $A^1, \dots, A^n$. These columns generate the n-plane $\Phi(\mathbb{R}^n)$ of the orbit $\mathcal{O}$. If there exists a nonzero integral vector $(k_1, \dots, k_N)$ orthogonal to $\mathcal{O}$ then it is orthogonal to the vectors $A^1, \dots, A^n$. This implies that $k_1 A_1 + \dots + k_N A_N = 0$, i.e. there exists a nontrivial linear combination with integral coefficients on the generators $A_1, \dots, A_N$ of $\mathcal{A}$, which gives a contradiction. By Appendix A the orbit $\mathcal{O}$ is dense in the torus $\mathbb{T}^N$. □

Lemma 4.2. *If the dimension of the orbit $\mathcal{O}$ is less than n, then the set V described by the T_{ij} has no isolated points, and the mean value M_Ω is equal to zero.*

Proof. If the dimension of the orbit is less than n, then $A^1, \dots, A^n$ are linearly dependent, and the solution space of the system $A^1 x_1 + \dots + A^n x_n = 0$ is nontrivial. If $T_{ij}(x) = 0$, then $T_{ij}(x+y) = 0$ for each y in the solution space. This means that V has no isolated points, and the mean value is equal to zero. □

If the orbit is $n$dimensional and $n < N$, then $\mathbb{R}^n$ is mapped bijectively to a dense n-dimensional orbit on the real torus $\mathbb{T}^N$ and trigonometric polynomials T_{ij} regarded as functions $\widetilde{T}_{ij}$ on $\mathbb{T}^N$ define a set $\widetilde{V}$. The intersection of the set $\widetilde{V}$ and the orbit $\mathcal{O}$ is the semitrigonometric set V. We are adding up the values of $\widetilde{T}$ over the isolated (in the topology of the orbit) points of intersection of the set $\widetilde{V}$ and the orbit. Here is the main result of this chapter which now follows directly from Theorem 3.4.

Theorem 4.3. *Let V be a quasiperiodic semitrigonometric subset of $\mathbb{R}^n$, T a quasiperiodic trigonometric polynomial. Let $\widetilde{V}$, $\widetilde{T}$, $\mathcal{O}$ be the corresponding set, function on $\mathbb{T}^N$, and orbit, as constructed above. Let Ω be a bounded subset of $\mathbb{R}^n$ with nonzero volume, whose boundary has fractal dimension less than n.*

Then the mean value M_Ω always exists and is equal to the $\mathcal{O}$-transversal $\widetilde{T}$-weighted volume of $\widetilde{V}$, assuming that the dimension of the orbit is n, and $n < N$. If the dimension of the orbit is less than n, the mean value is equal to zero.

Remark 4.4. If the orbit has the same dimension as the torus, i.e., $N = n$, the set $V \subset \mathbb{R}^n$ and the function T have n linearly independent periods $A^1, \dots, A^n$, which define a torus $\mathbb{R}^n/(\mathbb{Z}A^1 \oplus \dots \oplus \mathbb{Z}A^n)$. The mean value M_Ω in this case is equal to the sum of the values of T at the isolated points of V in this torus, divided by the volume of the torus.

Remark 4.5. This theorem holds and our proof works if V is a quasiperiodic subanalytic subset of $\mathbb{R}^n$, that is, V is the intersection of a subanalytic subset $\widetilde{V}$ of $\mathbb{T}^N$ with some dense n-dimensional orbit, and T is the restriction to this orbit of some bounded subanalytic function on $\widetilde{V}$.

Corollary 4.6. *If all the multiples $k\alpha$, $k \in \mathbb{Z} \setminus \{0\}$ of each frequency α of T do not belong to the subgroup $\mathcal{A}$ of $(\mathbb{R}^n, +)$ generated by the frequencies of T_{ij}, then the mean value M_Ω is equal to zero.*

Proof. It is enough to check this statement for T with a single frequency α, i.e., for $T(x) = c\cos 2\pi\alpha x + d\sin 2\pi\alpha x$. The set of all frequencies of the trigonometric polynomials T_{ij} describing the set V generates a group $\mathcal{A}'$ in $(\mathbb{R}^n, +)$, which is the direct sum of infinite cyclic subgroups:

$$\mathcal{A}' = (A_1) \oplus \cdots \oplus (A_{N-1}).$$

Since no multiple of the frequency α of T belongs to $\mathcal{A}'$, the group $\mathcal{A}$ obtained from $\mathcal{A}'$ by throwing in α, is the direct sum:

$$\mathcal{A} = (A_1) \oplus \cdots \oplus (A_{N-1}) \oplus (\alpha).$$

Therefore, raising the situation to the torus $\mathbb{R}^N/\mathbb{Z}^N$, we obtain a function $\widetilde{T}(\varphi_N) = c\cos\varphi_N + d\sin\varphi_N$ that depends only on the variable φ_N, and a set $\widetilde{V}$, which is described by the functions $\widetilde{T}_{ij}$ that do not depend on φ_N.

By Theorem 4.3, the mean value is equal to the $\mathcal{O}$-transversal $\widetilde{T}$-weighted volume of $\widetilde{V}$. Since the $\widetilde{T}_{ij}$ do not depend on φ_N, the manifold $M_{N-n}(\widetilde{V})$ (which is the smooth $N-n$ dimensional, transversal to the orbit $\mathcal{O}$, part of $\widetilde{V}$) is the direct product $\widetilde{V}' \times S^1$, where $\widetilde{V}'$ is a manifold in $\mathbb{R}^{N-1}/\mathbb{Z}^{N-1}$, and $S^1 = \mathbb{R}/\mathbb{Z}$. The form $\omega = p^* d\varphi_1 \dots d\varphi_N(\dots, A^1, \dots, A^n)$ can be written as $\omega_1 \wedge d\varphi_N$ where ω_1 is a form on $\widetilde{V}'$. Therefore, the mean value is equal to

$$\int_{M_{N-n}(\widetilde{V})} T\omega = \int_{\widetilde{V}'} \left(\int_{S^1} (c\cos 2\pi\varphi_N + d\sin 2\pi\varphi_N) d\varphi_N \right) \omega_1 = 0.$$

□

5. Zeros of systems of exponential sums

In this section we apply our main result (Theorem 3.4) to the complex case, that is, to computing the mean value of an exponential sum over the zeros of a system of n exponential sums in $\mathbb{C}^n$. We show that this mean value always exists. From the integral representation for the mean value that we obtain, we deduce a proof of Conjecture 2.3 in the case when the frequencies of the exponential sum are not commensurate with the frequencies of the system.

5.1. **Definitions.** Fix a system

$$F_1(z) = \cdots = F_n(z) = 0, \quad z \in \mathbb{C}^n \tag{5.1}$$

of exponential sums with a developed collection of Newton polyhedra. Let $\mathcal{F}$ be a family of systems of n exponential sums with the same collection of spectra as (5.1), and whose coefficients have the same absolute values as the corresponding coefficients in (5.1). The following theorem is a particular case of a result proved by Gelfond [7, 8].

Theorem 5.1. *There exists $R > 0$ such that all the zeros of the systems from the family $\mathcal{F}$ belong to the strip $S_R \times \operatorname{Im}\mathbb{C}^{\mathrm{n}}$, where $S_R \subset \operatorname{Re}\mathbb{C}^{\mathrm{n}}$ is a ball of radius R centered at the origin. This implies that all the zeros of the systems from $\mathcal{F}$ are isolated.*

Let G be an exponential sum with real frequencies, and Ω a bounded subset of $\operatorname{Im}\mathbb{C}^n$ with nonzero volume. For $\lambda > 0$, let $S_\Omega(\lambda)$ be the sum of the values of G at the zeros of (5.1) (counting multiplicities) that belong to the strip $\mathbb{R}^n \times \lambda\Omega \subset \mathbb{C}^n$. Let $S_\Omega(\lambda)^{\geq k}$ be the sum of the values of G at the zeros (not counting multiplicities) of the system (5.1) of multiplicity at least k that belong to the strip $\mathbb{R}^n \times \lambda\Omega \subset \mathbb{C}^n$. By Theorem 5.1, $S_\Omega(\lambda)$ and $S_\Omega(\lambda)^{\geq k}$ are well–defined.

Define

$$M_\Omega = \lim_{\lambda\to\infty} \frac{S_\Omega(\lambda)}{\operatorname{Vol}(\lambda\Omega)}, \quad M_\Omega^{\geq k} = \lim_{\lambda\to\infty} \frac{S_\Omega^{\geq k}(\lambda)}{\operatorname{Vol}(\lambda\Omega)}.$$

Theorem 5.2 below provides an integral formula for computing the mean values M_Ω, $M_\Omega^{\geq k}$ similar to the formula in the real case (Theorem 4.3). In particular, Theorem 5.2 implies that the mean value M_Ω always exists. Another consequence of this integral formula is that in the case when the frequencies of G are not commensurate with the frequencies of the exponential polynomials from the system, the mean value M_Ω is equal to zero (Corollary 5.3).

5.2. **Averaging an exponential sum over the zeros of a system of exponential sums.** As in the real case, we deduce an integral formula for the mean value from Theorem 3.4. The exponential sums F_j, G are quasiperiodic along the imaginary subspace of $\mathbb{C}^n$. Since the system (5.1) has a developed collection of Newton polyhedra, the real parts of the zeros of this system belong to some ball $S_R \subset \operatorname{Re}\mathbb{C}^n$. Therefore, we can restrict our attention to this ball, and think of the F_j and G as of functions quasiperiodic along the imaginary subspace and periodic along the real subspace. This allows us to consider $\mathbb{C}^n$ as an orbit on some real torus and apply Theorem 3.4 to this situation.

As before, the set of all frequencies α of the exponential sums F_j, G generates a subgroup $\mathcal{A}$ in $(\mathbb{R}^n, +)$, which is a direct sum of infinite cyclic subgroups:

$$\mathcal{A} = (\alpha_1) \oplus \cdots \oplus (\alpha_N).$$

In other words, there exist $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^n$ with no integral relations, such that each of the frequencies $\alpha \in \mathcal{A}$ is an integral combination of $\alpha_1, \ldots, \alpha_N$.

Define a linear map

$$\Phi_1 : \operatorname{Im}\mathbb{C}^n \to \mathbb{R}^N \quad \text{by} \quad \Phi_1(y) = (\alpha_1 y, \ldots, \alpha_N y), \quad y \in \operatorname{Im}\mathbb{C}^n.$$

According to Theorem 5.1 the real parts of the zeros of the system (5.1) belong to the open ball $S_R \subset \operatorname{Re}\mathbb{C}^n$ centered at the origin. Let $\Phi_0 : \operatorname{Re}\mathbb{C}^n \to \mathbb{R}^n$ be a composition of a shift and a rescaling that maps S_R inside the unit cube. We now think of F_j and G as of functions periodic along the real subspace.

Then $\Phi = \Phi_0 \times \Phi_1$ maps $\mathbb{C}^n = \mathrm{Re}\ \mathbb{C}^n \times \mathrm{Im}\ \mathbb{C}^n$ to $\mathbb{R}^n \times \mathbb{R}^N$. The map Φ defines an action of $\mathbb{C}^n$ on $\mathbb{T}^{n+N}$ by

$$\varphi \mapsto (\varphi + \Phi(x,y)) \bmod \mathbb{Z}^{n+N}, \quad \varphi \in \mathbb{T}^{n+N}, \quad x \in \mathrm{Re}\ \mathbb{C}^n, \quad y \in \mathrm{Im}\ \mathbb{C}^n.$$

Let $\mathcal{O}$ be the orbit of this action through the origin. Since each of the frequencies α is an integral combination of $\alpha_1, \dots, \alpha_n$, the sums F_j, G are the restrictions to $\mathcal{O}$ of some functions $\widetilde{F}_j$, $\widetilde{G}$ defined on $\mathbb{T}^{n+N}$. We note that these functions may only be non-analytic at the points φ where at least one of the coordinates $\varphi_1, \dots, \varphi_n$ is equal to zero.

The set of all zeros of the system (5.1) of multiplicity at least k is an analytic set defined by exponential equations $F_j(z) = 0$ and a few more exponential equations whose frequencies belong to the subgroup $\mathcal{A}$ of $(\mathbb{R}^n, +)$ (see, for example, [4]).

Let $V^{\geq k}$ be the set of all zeros of the system of multiplicity greater than or equal to k. Consider the system of exponential sums that defines $V^{\geq k}$. Regarded as functions on $\mathbb{T}^{n+N}$, these sums define a subset $\widetilde{V}^{\geq k}$ of the torus $\mathbb{T}^{n+N}$. We now state the main result of this section.

Theorem 5.2. *Consider a system of exponential sums with real frequencies*

$$F_1(z) = \dots = F_n(z) = 0, \quad z \in \mathbb{C}^n \tag{5.2}$$

with a developed collection of Newton polyhedra. Let G be an exponential sum with real frequencies, and Ω a bounded subset of $\mathbb{R}^n$ with nonzero volume whose boundary has fractal dimension less than n. Let $\widetilde{V}^{\geq k}$, $\widetilde{G}$, $\mathcal{O}$ be the corresponding subsets of $\mathbb{T}^{n+N}$, function on $\mathbb{T}^{n+N}$, and orbit, as constructed above.

Then if $n < N$ the mean value $M_\Omega^{\geq k}$ is equal to the $\widetilde{G}$-weighted $\mathcal{O}$-transversal volume of $\widetilde{V}^{\geq k}$. The mean value M_Ω is equal to the sum $M_\Omega^{\geq 1} + \dots + M_\Omega^{\geq k}$ for some k.

Proof. We first show that the dimension of the orbit $\mathcal{O}$ is $2n$. It is enough to show that the dimension of the plane $\Phi_1(\mathrm{Im}\ \mathbb{C}^n)$ is n. This plane is generated by the columns of the matrix of the linear map Φ_1. The rows of this matrix are the generators $\alpha_1, \dots, \alpha_N$ of the group $\mathcal{A}$. Thus the dimension of the plane $\Phi_1(\mathrm{Im}\ \mathbb{C}^n)$ is equal to the dimension of the linear space generated by the frequencies $\alpha_1, \dots, \alpha_N$. Since the collection of the Newton polyhedra of the system (5.2) is developed, its frequencies generate $\mathbb{R}^n$. We conclude that the dimension of the orbit is n.

Similarly to the real case, it is easy to see that there are no integral vectors orthogonal to the orbit $\mathcal{O}$. Therefore, the orbit $\mathcal{O}$ is dense in the torus $\mathbb{T}^{n+N}$ (see Lemma 4.1).

To derive Theorem 5.2 from Theorem 3.4 we show that $\widetilde{V}^{\geq k}$ is an analytic subset of $\mathbb{T}^{n+N}$ and $\widetilde{G}$ is analytic on $\widetilde{V}^{\geq k}$. Each of $\widetilde{V}^{\geq k}$ is defined by a system of equations analytic everywhere on $\mathbb{T}^{n+N}$ but possibly at the points where at least one of the coordinates $\varphi_1, \dots, \varphi_n$ is equal to zero. Notice that $\widetilde{V}^{\geq k}$ is contained in the "cylinder" $\Phi_0(S_R) \times \mathbb{T}^N$. Indeed, the intersections of $\widetilde{V}$ with the orbit through the

origin belong to this cylinder since the zeros of the initial system lie in the strip $S_R \times \operatorname{Im} \mathbb{C}^n$. Restricting the functions $\widetilde{F}_j$ to a shifted orbit we obtain functions that differ from F_j by a shift along the imaginary subspace. By Theorem 5.1 the zeros of this new system belong to the strip $S_R \times \operatorname{Im} \mathbb{C}^n$, and we conclude that $\widetilde{V}^{\geq k} \subset \Phi_1(S_R) \times \mathbb{T}^N$. Therefore, $\widetilde{V}^{\geq k}$ is locally defined by a system of analytic equations, and G is analytic on $\widetilde{V}^{\geq k}$.

To show that M_Ω can be computed by the formula $M_\Omega = M_\Omega^{\geq 1} + \cdots + M_\Omega^{\geq k}$ for some k, we need to prove that the sets $V^{\geq k}$ are empty, starting from some k. Assume that there exists a sequence $z_k = x_k + iy_k$, $x_k \in S_R$, such that z_k belongs to $V^{\geq k}$. Raising this sequence to $S_R \times \mathbb{T}^N$, we obtain a sequence $\widetilde{z}_k$ in $S_R \times \mathbb{T}^N$ such that $\widetilde{z}_k$ belongs to $\widetilde{V}^{\geq k}$. Since $S_R \times \mathbb{T}^N$ is compact, there exists a limit point $\widetilde{z}_0 \in S_R \times \mathbb{T}^N$. This point $\widetilde{z}_0$ belongs to each $\widetilde{V}^{\geq k}$. Although this point does not necessarily belong to the orbit $S_R \times \Phi_1(\operatorname{Im} \mathbb{C}^N/\mathbb{Z}^N)$, it belongs to some shifted orbit. The restrictions of the functions $\widetilde{F}_j$ to this shifted orbit define an exponential system whose zeros are isolated and belong to $S_R \times \operatorname{Im} \mathbb{C}^n$ (Theorem 5.1). Therefore, $\widetilde{z}_0$ is a zero of the shifted system of infinite multiplicity, which is impossible since all the zeros of this system are isolated. We proved that all $V^{\geq k}$ are empty starting from some k.

Now the theorem follows from Theorem 3.4. □

The proof of the following corollary is a repetition of the proof of Corollary 4.6.

Corollary 5.3. *If all the multiples $k\alpha$, $k \in \mathbb{Z} \setminus \{0\}$ of each frequency α of F do not belong to the subgroup $\mathcal{A}$ of $(\mathbb{R}^n, +)$ generated by the frequencies of the F_j, then the mean value M_Ω is equal to zero.*

6. Proof of Theorem 3.4

Proof. Recall that $\mathcal{O} = (a + \Phi(\mathbb{R}^n)) \bmod \mathbb{Z}^N$, where $a \in \mathbb{T}^N$, and $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ is a linear injective map. We send the torus $\mathbb{T}^N$ to the cube $I = [0,1)^N \subset \mathbb{R}^N$ by cutting $\mathbb{T}^N$ along $\varphi_j = 1$. Choose a new basis $\{B^1, \ldots, B^{N-n}, A^1, \ldots, A^n\}$ in $\mathbb{R}^N$ whose last n coordinate vectors are the images of the standard basis in $\mathbb{R}^n$ under Φ, $A^j = \Phi(e_j)$. Note that the vectors $A^1, \ldots, A^n$ generate the plane of the orbit $\Phi(\mathbb{R}^n)$. Let the remaining vectors $B^1, \ldots, B^{N-n}$ form a basis of the $(N-n)$-plane L orthogonal to the orbit, so that the determinant of the change of coordinates matrix from the standard basis in $\mathbb{R}^N$ to this new basis is equal to one.

Let $x = (x_1, \ldots, x_n)$ be the coordinates along the orbit, and $y = (y_1, \ldots, y_{N-n})$ the coordinates in the orthogonal plane L. We choose a cube $Q = Q_y \times Q_x$ whose edges go along the new coordinate axes and $I \subset Q$. We now regard $\widetilde{V}$ as a subset of $I \subset Q$ and $\widetilde{T}(\varphi)$ as a function $\widetilde{T}(y, x)$ on $\widetilde{V} \subset I$.

The set $\widetilde{V}$ is a subset of Q, subanalytic in $\mathbb{R}P^N$, and the graph of the function $\widetilde{T}$ is a subset of $Q \times \mathbb{R}$ subanalytic in $\mathbb{R}P^{N+1}$. Therefore, by the cell decomposition theorem (Theorem 8.3), there exists a decomposition of $\mathbb{R}^N$ with the fixed order of

coordinates (y, x) into finitely many cells such that $\widetilde{V}$ is a union of cells and $\widetilde{T}$ is C^1 on each of these cells.

Since the cell decomposition respects the standard projection, each cell is either a part of the closure of another cell, or is disjoint from it. Only $(i_1, \dots, i_{N-n}, 0, \dots, 0)$-cells that do not belong to the closure of a cell of higher dimension may have isolated intersections with the orbit. We will show later that cells of dimension less than $N-n$ do not contribute to the mean value. Therefore, we can restrict our attention to $(1, \dots, 1, 0, \dots, 0)$-cells with $N-n$ ones and n zeros that do not belong to a closure of a cell of higher dimension.

Let C be one of such cells. Then

$$C = \{(y, x) \mid y \in D,\ x = H(y)\},$$

where $H = (H_1, \dots, H_n)$ for some C^1 functions $H_1, \dots, H_n$, and D is a $(1, \dots, 1)$-cell in Q_y.

Now we define a counting function g on the torus $\mathbb{T}^N$. Through each point $a \in C$ we draw an n-plane parallel to the orbit. If φ belongs to $(a + \Phi(B_r))/\mathbb{Z}^N$, where $B_r \subset \mathbb{R}^n$ is the ball of radius r centered at the origin, we define $g(\varphi) = \widetilde{T}(a)/v_r$, where v_r is the volume of the ball $B_r \subset \mathbb{R}^n$. (Here r is chosen so small that the mapping $x \mapsto \Phi(x)/\mathbb{Z}^N$ restricted to B_r is one-to-one.) Otherwise we define $g(\varphi) = 0$.

In (y, x)-coordinates, g is defined by

$$g(y, H(y) + x) = \begin{cases} \widetilde{T}(y, H(y))/v_r, & \text{if } y \in D,\ x \in B_r \\ 0, & \text{otherwise.} \end{cases}$$

Figure 2 illustrates the definition of g in the case when the cell C is one-dimensional. Through each $a \in C$ draw a vertical segment of length $2r$ centered at a. This segment is the ball $a + B_r$ from the general construction. Define g to be identically equal to $\widetilde{T}(a)/2r$ on that segment.

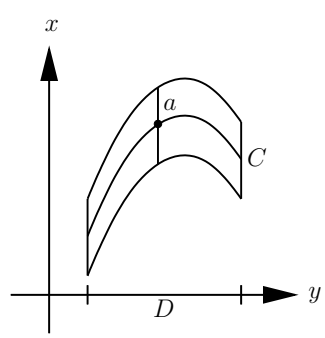

FIGURE 2. Definition of the counting function g

Since the function $\widetilde{T}$ is continuous and bounded on C and the functions defining C are continuously differentiable, g is Riemann integrable on the torus $\mathbb{T}^N$. We apply Weyl's equidistribution law for multidimensional trajectories (Theorem 7.1):

$$\lim_{\lambda\to\infty}\frac{1}{\mathrm{Vol}(\lambda\Omega)}\int_{\lambda\Omega} g(\Phi(x))dx = \int_{\mathbb{T}^N} g(\varphi)d\varphi. \tag{6.1}$$

Let us show first that the left hand side part of this identity is exactly the contribution of the cell C to the mean value M_Ω. If $a+B_r$, a ball centered at the point $a\in\mathbb{R}^n$, lies entirely in $\lambda\Omega$ then by the definition of g

$$\int_{a+B_r} g(\Phi(x))dx = T(a).$$

It remains to take care of the points a such that $a+B_r$ does not lie entirely in $\lambda\Omega$. The dimension of the boundary of Ω is less than or equal to $n-\delta$ for some $\delta>0$. Hence the number of balls of radius r that are required to cover the r-neighborhood of the boundary of $\lambda\Omega$ (i.e. the set of all points whose distance to the boundary is less than r) is bounded by $c\lambda^{n-\delta}$ for some constant c. The number of isolated points of V in each of such balls is bounded by the number of cells in our cell decomposition. Therefore, the result of adding up T over the isolated points of V, whose distance to the boundary of $\lambda\Omega$ is less than r, is bounded by $c_0\lambda^{n-\delta}$ for some constant c_0. When we divide the result by the volume of $\lambda\Omega$, which is proportional to λ^n, the quotient approaches 0 as λ approaches infinity. Therefore, we can disregard the points that are close to the boundary of $\lambda\Omega$ while computing the mean value M_Ω. We have proved that

$$\lim_{\lambda\to\infty}\frac{1}{\mathrm{Vol}(\lambda\Omega)}\int_{\lambda\Omega} g(\Phi(x))dx$$

is the contribution of the cell C to the mean value M_Ω.

We now work on the rewriting of the right hand side part of (6.1).

$$\begin{aligned}\int_{\mathbb{T}^N} g(\varphi)d\varphi &= \int_Q g(y,x)dydx = \int_{Q_y}\left(\int_{Q_x} g(y,x)dx\right)dy\\ &= \int_D \widetilde{T}(y,H(y))dy = \int_C \widetilde{T}(\varphi)p^*d\varphi(\dots,A^1,\dots,A^n) = \int_C \widetilde{T}(\varphi)p^*\omega.\end{aligned}$$

Next we note that cells of dimension less than $N-n$ do not contribute to the mean value. Indeed, for each of such cells we construct a counting function g and repeat the above argument. It follows that the contribution of each of such cells is

$$\int_{\mathbb{T}^N} g(\varphi)d\varphi,$$

which is equal to zero since g is zero on a set of full measure in $\mathbb{T}^N$.

The union of all $(i_1,\dots,i_{N-n},0,\dots,0)$-cells that do not belong to the closure of a cell of higher dimension is the $(N-n)$-dimensional manifold $M_{N-n}(\widetilde{V})$, up

to possibly a few cells of smaller dimension. Those cells of smaller dimension do not contribute to the mean value. We conclude that the mean value is equal to the $\widetilde{T}$-weighted $\mathcal{O}$-transversal volume of $\widetilde{V}$. □

7. Appendix A: Weyl's equidistribution law for dense multidimensional orbits on the real torus.

Classical Weyl's equidistribution law [16] states that the time-average of a Riemann integrable function along a one-dimensional dense orbit on a real torus coincides with the space average. Here we state a version of Weyl's law for multidimensional orbits.

Let $x \in \mathbb{R}^n$, $\varphi \in \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. A linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ defines an action of $\mathbb{R}^n$ on $\mathbb{T}^N$ by $\varphi \mapsto (\varphi + \Phi(x)) \bmod \mathbb{Z}^N$. Let $\mathcal{O}$ be an orbit of this action.

Theorem 7.1. *Let f be a Riemann integrable function on the torus $\mathbb{T}^N$, $\mathcal{O}$ an orbit on $\mathbb{T}^N$ through an arbitrary point $\varphi_0 \in \mathbb{T}^N$ such that there are no nonzero integral vectors orthogonal to $\mathcal{O}$, and Ω a subset of $\mathbb{R}^n$ with nonzero volume.*

Then

$$\frac{1}{\mathrm{Vol}(\lambda\Omega)} \int_{\lambda\Omega} f(\varphi_0 + \Phi(x))dx \to \int_{\mathbb{T}^N} f(\varphi)d\varphi, \quad as \quad \lambda \to \infty.$$

The proof of this theorem in [12] is a direct generalization of Weyl's original argument for one-dimensional orbits.

Remark 7.2. This theorem implies that an orbit is dense if and only if there are no nonzero integral vectors orthogonal to the plane of the orbit.

8. Appendix B: Cell-decomposition theorem

The cell-decomposition theorem for subanalytic sets states that a subanalytic set can be partitioned into finitely many cells, which are subanalytic subsets of especially simple form. We will follow the formulation of the cell-decomposition theorem as presented in [2].

Definition 8.1. Let $(i_1, \dots, i_n)$ be a sequence of zeros and ones of length n. An $(i_1, \dots, i_n)$-*cell* is a subset of $\mathbb{R}^n$ obtained by induction on n as follows:

(i) a (0)-cell is a point in $\mathbb{R}$, a (1)-cell is an interval $(a, b) \in \mathbb{R}$;

(ii) suppose $(i_1, \dots, i_n)$-cells are already defined; then an $(i_1, \dots, i_n, 0)$-cell is the graph $\Gamma(f)$ of a C^1 function f on an $(i_1, \dots, i_n)$-cell provided that $\Gamma(f) \subset \mathbb{R}^{n+1}$ is subanalytic in $\mathbb{R}P^{n+1}$; further, an $(i_1, \dots, i_n, 1)$-cell is a set

$$(f, g)_X := \{(x, r) \in X \times \mathbb{R} : f(x) < r < g(x)\},$$

where X is an $(i_1, \dots, i_n)$-cell, functions f, g are C^1 functions on X whose graphs $\Gamma(f), \Gamma(g) \subset \mathbb{R}^{n+1}$ are subanalytic in $\mathbb{R}P^{n+1}$, and $f < g$ on X. The constant functions $f(x) = -\infty$, $g(x) = +\infty$ are also allowed.

Definition 8.2. A *decomposition* of $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into finitely many cells. The definition is by induction on n:

(i) a decomposition of $\mathbb{R}$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\}$$

where $a_1 < \dots < a_k$ are points in $\mathbb{R}$.

(ii) a decomposition of $\mathbb{R}^{n+1}$ is a finite partition into cells such that the set of projections of these cells to the first n coordinates forms a decomposition of $\mathbb{R}^n$.

Theorem 8.3. (The cell decomposition theorem) *Let X be a subset of $\mathbb{R}^n$ subanalytic in $\mathbb{R}P^n$. Let f be a function on X whose graph in $\mathbb{R}^{n+1}$ is a subanalytic subset of $\mathbb{R}P^{n+1}$. Then there exists a (finite) decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that X is a union of cells in $\mathcal{D}$ and the restriction $f|_B : B \to \mathbb{R}$ to each cell $B \in \mathcal{D}$ is C^1.*

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