THE DYNAMICAL MOTION OF THE ZEROS OF THE PARTIAL SUMS OF e^z, AND ITS RELATIONSHIP TO DISCREPANCY THEORY*

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Dedicated to Edward B. Saff on his 64th birthday, January 2, 2008.

Abstract. With $s_n(z) := \sum_{k=0}^{n} z^k/k!$ denoting the n-th partial sum of e^z, let its zeros be denoted by $\{z_{k,n}\}_{k=1}^{n}$ for any positive integer n. If θ_1 and θ_2 are any angles with $0 < \theta_1 < \theta_2 < 2\pi$, let Z_{θ_1,θ_2} be the associated sector, in the z-plane, defined by

$$Z_{\theta_1,\theta_2} := \{z \in \mathbb{C} : \theta_1 \leq \arg z \leq \theta_2\}.$$

If $\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)$ represents the number of zeros of $s_n(z)$ in the sector Z_{θ_1,θ_2}, then Szegő showed in 1924 that

$$\lim_{n\to\infty} \frac{\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)}{n} = \frac{\phi_2 - \phi_1}{2\pi},$$

where ϕ_1 and ϕ_2 are defined in the text. The associated *discrepancy function* is defined by

$$\mathrm{disc}_n(\theta_1,\theta_2) := \#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right) - n\left(\frac{\phi_2 - \phi_1}{2\pi}\right).$$

One of our new results shows, for any θ_1 with $0 < \theta_1 < \pi$, that

$$\mathrm{disc}_n(\theta_1, 2\pi - \theta_1) \sim K \log n, \text{ as } n \to \infty,$$

where K is a positive constant, depending only on θ_1. Also new in this paper is a study of the *cyclical nature* of $\mathrm{disc}_n(\theta_1,\theta_2)$, as a function of n, when $0 < \theta_1 < \pi$ and $\theta_2 = 2\pi - \theta_1$. An upper bound for the approximate cycle length, in this case, is determined in terms of ϕ_1. All this can be viewed in our *Interactive Supplement*, which shows the dynamical motion of the (normalized) zeros of the partial sums of e^z and their associated discrepancies.

Key words. partial sums of e^z, Szegő curve, discrepancy function

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