Minimal Geršgorin Sets and $\omega$-Matrices

G. M. ENGEL
International Business Machines, 10 IBM 68, Owego, N.Y. 13827

and

R. S. VARGA
Department of Mathematics, Kent State University, Kent, OH 44242
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G. M. ENGEL†
*International Business Machines, 10 IBM 68, Owego, N.Y. 13827*

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R. S. VARGA‡
*Department of Mathematics, Kent State University, Kent, OH 44242*

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If $G(C)$ denotes the minimal Geršgorin set for $C \in \mathbb{C}^{n \times n}$, and if, for any nonempty subset $\alpha$ of the first $n$ positive integers, $C[\alpha]$ denotes the principal minor of $C$ determined by $\alpha$, then conditions are determined which characterize matrices $A$ and $B$ in $\mathbb{C}^{n \times n}$ such that the inclusions $G((D + B)[\alpha]) \subseteq G((D + A)[\alpha])$ are valid for all subsets $\alpha$ of the first $n$ positive integers, and for all diagonal matrices $D$ in $\mathbb{C}^{n \times n}$. Connections with the newly defined set of $\omega$-matrices are also included.

1. INTRODUCTION

For any matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, let $G(A)$ denote its minimal Geršgorin set (cf. [6]), i.e.,

$$G(A) = \bigcap_{x \in \mathbb{C}^n} \left\{ \sum_{i=1}^{n} |a_{i,j}|x_i \right\}$$

$$\sum_{j \neq i} |x_j|$$

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where \( \mathbb{R}_+^n \) denotes the collection of column vectors \( x = [x_1, x_2, \ldots, x_n]^T \) in \( \mathbb{R}^n \) with \( x_i > 0 \) for all \( 1 \leq i \leq n \), and where the sum in (1.1) is defined to be zero when \( n = 1 \). With this notation, we seek conditions on matrices \( A = [a_{i,j}] \) and \( B = [b_{j,i}] \) in \( \mathbb{C}^{n,n} \) which insure that
\[
\{ G((D + B)[x]) \leq G((D + A)[x]) \text{ for all } \phi \neq x \leq \langle n \rangle = \{1, 2, \ldots, n\} \}
\]
and for all \( D = \text{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{C}^{n,n} \), (1.2)
where \( A[x] \) in general denotes the principal submatrix of \( A \) determined by \( x \), i.e., \( A[x] = [a_{i,j}] \) where \( i, j \in x \). Our main result, Theorem 6, gives two conditions on \( A \) and \( B \), each of which is equivalent with (1.2).

On considering (1.2), we first observe from (1.1) that, on choosing \( x = \{i\} \) for any \( i \in \langle n \rangle \), the inclusion of (1.2) implies that
\[
a_{i,i} = b_{i,i} \text{ for all } i \in \langle n \rangle. \tag{1.3}
\]
Next, we also observe from (1.1) that the off-diagonal entries of \( A \) enter into the definition of the minimal Geršgorin set, \( G(A) \), only through their moduli. This suggests that conditions which insure (1.2) will similarly depend only on the moduli of the off-diagonal entries of \( A \) and \( B \).

2. NOTATION AND TERMINOLOGY

In this section, we collect some needed notation, terminology, and background material on minimal Geršgorin sets and \( \omega \)-matrices. To begin, assumed that \( A = [a_{i,j}] \in \mathbb{C}^{n,n} \) is reducible (cf. [5, p. 46]). It is well known that there is a permutation matrix \( P \in \mathbb{R}^{n,n} \) for which
\[
\begin{array}{c}
\text{PAP}^T = \\
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\end{array}
\tag{2.1}
\]
where each \( A_{j,j} \), \( 1 \leq j \leq r \), is irreducible. Here, for convenience, all \( 1 \times 1 \) matrices are defined to be irreducible. Then, it is known (cf. [6, p. 725]) that the minimal Geršgorin set for \( A \) is precisely the union of the minimal Geršgorin sets for \( A_{j,j} \), \( 1 \leq j \leq r \), i.e.,
\[
G(A) = \bigcup_{i=1}^{r} G(A_{i,i}). \tag{2.2}
\]

Next, consider any real matrix \( A = [a_{i,j}] \in \mathbb{R}^{n,n} \) satisfying
\[
a_{i,i} \text{ real; } a_{i,j} \geq 0, \quad i \neq j; \text{ for all } i, j \in \langle n \rangle. \tag{2.3}
\]
Such matrices are called essentially nonnegative (cf. [1], [5]). As a consequence of the Perron–Frobenius theory of nonnegative matrices, such a matrix satisfying (2.3) possesses a real eigenvalue \( \nu \), i.e., \( \nu \in \text{spec} (A) \): \( \{ \lambda : \det (A - \lambda I) = 0 \} \), which satisfies
\[
\nu \geq \text{Re} \lambda, \quad \text{for all} \quad \lambda \in \text{spec} (A),
\]
and the inclusions
\[
\min_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \} \leq \nu \leq \max_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \}, \quad \text{for all} \quad x \in \mathbb{R}_+^n. \tag{2.5}
\]

Furthermore, \( \nu \) is characterized (cf. [3, p. 201], [4]) by
\[
\nu = \inf_{x \in \mathbb{R}_+^n} \{ \max_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \} \}, \tag{2.6}
\]
and, if \( A \) is irreducible,
\[
\nu = \sum_{j \neq \langle n \rangle} a_{i,j} x_j / x_i \quad \text{for all} \quad i \in \langle n \rangle, \quad \text{for some} \quad x \in \mathbb{R}_+^n. \tag{2.7}
\]

These facts can be used as follows. For any \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \) and for any \( z \in \mathbb{C} \), define the real matrix \( T(z) = [t_{i,j}(z)] \in \mathbb{R}^{n \times n} \) by
\[
t_{i,j}(z) = -|z - a_{i,j}|; \quad t_{i,j}(z) = |a_{i,j}|, \quad i \neq j; \quad \text{for all} \quad i, j \in \langle n \rangle, \tag{2.8}
\]
so that \( T(z) \) satisfies (2.3). Denoting the associated eigenvalue of \( T(z) \), satisfying (2.4)–(2.6), by \( v(z; A) \), then \( v(z; A) \) satisfies, from (2.5) and (2.6),
\[
\min_{i \in \langle n \rangle} \{ -|z - a_{i,i}| + \sum_{j \neq \langle n \rangle} |a_{i,j}| x_j / x_i \} \leq v(z; A)
\]
\begin{equation}
\leq \max_{i \in \langle n \rangle} \{ -|z - a_{i,i}| + \sum_{j \neq \langle n \rangle} |a_{i,j}| x_j / x_i \}, \quad \text{for all} \quad x \in \mathbb{R}_+^n, \tag{2.9}
\end{equation}
and
\[
v(z; A) = \inf_{x \in \mathbb{R}_+^n} \{ \max_{i \in \langle n \rangle} \{ -|z - a_{i,i}| + \sum_{j \neq \langle n \rangle} |a_{i,j}| x_j / x_i \} \}. \tag{2.10}
\]

As proved in [6], the points of \( G(A) \) can then be characterized in terms of the function \( v(z; A) \) by means of

**Proposition 1** For any \( A \in \mathbb{C}^{n \times n} \), \( z \in G(A) \) iff \( v(z; A) \geq 0 \).

We also include from [6] the following result.

**Proposition 2** For any irreducible \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), then \( z \in \partial G(A) \), i.e., \( z \) is a boundary point of \( G(A) \), implies that there exists an \( x \in \mathbb{R}_+^n \) such that
\[
|z - a_{i,i}| = \sum_{j \neq \langle n \rangle} |a_{i,j}| x_j / x_i, \quad \text{for all} \quad i \in \langle n \rangle. \tag{2.11}
\]

Next, following Engel and Schneider [2], a matrix \( A \in \mathbb{C}^{n \times n} \) is called an \( \omega \)-matrix if
\[
\text{spec} (A[x]) \cap \mathbb{R} \neq \emptyset, \quad \text{for all} \quad x \quad \text{with} \quad \phi \neq x \subseteq \langle n \rangle, \tag{2.12}
\]
and, on defining

\[ l(A[x]) := \min \{ \text{spec}(A[z]) \cap \mathbb{R} \}, \]  

(2.13)

if, for arbitrary subsets \( \alpha \) and \( \beta \) of \( \langle n \rangle \), one has the nesting property

\[ \phi \neq \alpha \subseteq \beta \subseteq \langle n \rangle \quad \text{implies} \quad l(A[\beta]) \leq l(A[\alpha]). \]  

(2.14)

If \( \omega_{\omega} \) denotes the set of all \( \omega \)-matrices in \( \mathbb{C}^{n,n} \), then, as stated in [2], \( \omega_{\omega} \) contains all the Hermitian matrices, \( M \)-matrices, as well as all totally non-negative matrices, in \( \mathbb{C}^{n,n} \).

Continuing, (cf. [2]), a reflexive and transitive relation can be defined on \( \omega_{\omega} \) as follows: if \( A, B \in \omega_{\omega} \), then \( A \triangleright B \) if

\[ l(A[x]) \geq l(B[x]), \quad \text{for all} \quad x \quad \text{with} \quad \phi \neq \alpha \subseteq \langle n \rangle. \]  

(2.15)

In the case that \( A \) and \( B \) are Hermitian matrices in \( \mathbb{C}^{n,n} \), and thus elements of \( \omega_{\omega} \), it is easy to verify that \( A \triangleright B \) if \( A \) is positive semi-definite, and that \( A \triangleright B \) if \( A - B \) is positive semi-definite. In the same vein, we investigate implications of the relation \( A \triangleright B \) on a subset, \( \mathbb{Z}^{n,n} \), of \( \omega_{\omega} \) which contains the \( M \)-matrices of \( \omega_{\omega} \). Defining \( \mathbb{Z}^{n,n} \) as the subset of \( \mathbb{R}^{n,n} \) of all real matrices \( A = [a_{i,j}] \) for which

\[ a_{i,j} \text{ real; } a_{i,j} \leq 0, \quad i \neq j; \quad \text{for all} \quad i, j \in \langle n \rangle, \]  

(2.16)

then from (2.3), \( A \in \mathbb{Z}^{n,n} \) iff \( -A \) is essentially nonnegative. Thus, applying the characterizations of (2.4)-(2.7), it analogously follows that \( A = [a_{i,j}] \in \mathbb{Z}^{n,n} \) has a real eigenvalue \( l(A) \) such that

\[ l(A) = \Re \lambda, \quad \text{for all} \quad \lambda \in \text{spec}(A), \]  

(2.17)

\[ \min_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \} \leq l(A) \leq \max_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \}, \quad \text{for all} \quad x \in \mathbb{R}^n, \]  

(2.18)

\[ l(A) = \sup_{x \in \mathbb{R}^n} \{ \min_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \} \}, \]  

(2.19)

and if \( A \in \mathbb{Z}^{n,n} \) is irreducible, then

\[ l(A) = \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \quad \text{for all} \quad i \in \langle n \rangle, \quad \text{for some} \quad x \in \mathbb{R}^n. \]  

(2.20)

It is not difficult to verify (cf. [5, p. 30]) that \( l(A) \), so defined, also satisfies the nesting property of (2.14), so that \( \mathbb{Z}^{n,n} \) is a subset of \( \omega_{\omega} \). This brings us to the following useful characterization of \( \triangleright \) on \( \mathbb{Z}^{n,n} \).

**Proposition 3** Given \( A = [a_{i,j}] \) and \( B = [b_{i,j}] \) in \( \mathbb{Z}^{n,n} \), then \( A \triangleright B \) iff, for every \( x \) and \( y \) in \( \mathbb{R}^n \), and for every \( \phi \neq \alpha \subseteq \langle n \rangle \), there exist integers \( i \) and \( k \) in \( \alpha \) such that

\[ \sum_{j \in \alpha} a_{i,j} y_j / y_i \geq \sum_{j \in \alpha} b_{i,j} x_j / x_i. \]  

(2.21)

**Proof** First, assume that \( A \triangleright B \). It suffices to consider only the case \( \alpha = \langle n \rangle \), since the proof for any nonempty \( \alpha \) in \( \langle n \rangle \) is similar. By hypothesis,
\( l(A) \geq l(B) \) and hence, for any \( x \) and any \( y \) in \( \mathbb{R}_+^n \), we have from (2.18) that
\[
\max_{i \in \langle n \rangle} \left\{ \sum_{j \in \langle n \rangle} a_{i,j} y_j/y_i \right\} \geq l(A) \geq l(B) \geq \min_{i \in \langle n \rangle} \left\{ \sum_{j \in \langle n \rangle} b_{i,j} x_j/x_i \right\},
\]
which directly implies (2.21). Conversely, it follows from (2.21) that
\[
\max_{i \in \langle n \rangle} \left\{ \sum_{j \in \langle n \rangle} a_{i,j} y_j/y_i \right\} \leq \min_{i \in \langle n \rangle} \left\{ \sum_{j \in \langle n \rangle} b_{i,j} x_j/x_i \right\},
\]
for every \( x \) and \( y \) in \( \mathbb{R}_+^n \), and every \( \phi \neq \alpha \subseteq \langle n \rangle \). Hence, taking the supremum of the right side over \( x \) in \( \mathbb{R}_+^n \) gives, from (2.19),
\[
\sum_{j \in \langle n \rangle} a_{k,j} y_j/y_k \geq l(B[x]) \quad \text{for some} \quad k = k(y) \in \alpha, \quad \text{for all} \quad y \in \mathbb{R}_+^n.
\]
If \( A[x] \) is irreducible, we can, from (2.20), choose \( y \in \mathbb{R}_+^n \) so that
\[
\sum_{j \in \langle n \rangle} a_{k,j} y_j/y_k = l(A[x]) \quad \text{for all} \quad k \in \alpha, \quad \text{whence} \quad l(A[x]) \geq l(B[y]).
\]
If \( A[x] \) is however reducible, there is a subset \( \beta \) with \( \phi \neq \beta \subseteq \alpha \) for which \( A[\beta] \) is irreducible, and for which \( l(A[\beta]) = l(A[x]) \). Thus, with the above irreducible case,
\[
l(A[x]) = l(A[\beta]) \geq l(B[\beta]) \geq l(B[z]), \tag{2.22}
\]
the last inequality following from the resting property of (2.14), and (i) implies (i). \( \blacksquare \)

3. MAIN RESULT

We begin with

**Definition 4**: Given \( A = [a_{i,j}] \) and \( B = [b_{i,j}] \) in \( \mathbb{C}^{n,n} \), then \( A \) dominates \( B \) if, for every \( \phi \neq \alpha \subseteq \langle n \rangle \) for which \( B[x] \) is irreducible, and for every \( x \) and every \( y \) in \( \mathbb{R}_+^n \), there is an \( i \in \alpha \) for which
\[
\sum_{j \in \alpha} |a_{i,j}| y_j/y_i \geq \sum_{j \in \alpha} |b_{i,j}| x_j/x_i. \tag{3.1}
\]

Note that (3.1) of Definition 4 is a condition like that of (2.21) of Proposition 3, but differs essentially in that (3.1) holds for the *same* \( i \) in the sums of (3.1), while (2.21) holds for possibly different \( i \) and \( k \) in the sums of (2.21).

We note from Definition 4 that \( |A| \) dominates \( |B| \) iff, for any non-singular diagonal matrices \( S = \text{diag}[s_1, s_2, \ldots, s_n] \) and \( T = \text{diag}[t_1, t_2, \ldots, t_n] \), then \( |S^{-1}AS| \) dominates \( |T^{-1}BT| \). As another characterization in terms of \( \omega \)-matrices, we have

**Proposition 5**: Given \( A = [a_{i,j}] \) and \( B = [b_{i,j}] \) in \( \mathbb{C}^{n,n} \), then \( A \) dominates \( B \) iff \( D - |B| \geq |D - |A|| \) for all real \( D = \text{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n,n} \).

**Proof**: Assuming first that \( |A| \) dominates \( |B| \), consider any real \( D = \text{diag}[d_1, d_2, \ldots, d_n] \) in \( \mathbb{R}^{n,n} \) and any nonempty \( \alpha \subseteq \langle n \rangle \). If \( B[\alpha] \) is irreducible,
then, for every $x$ and $y$ in $\mathbb{R}^n_+$, it follows from (3.1) of Definition 4 that there is an $i \in \pi$ for which
\[ d_i - \sum_{j \in \pi} |b_{i,j}| x_j / x_i \geq d_i - \sum_{j \in \pi} |a_{i,j}| y_j / y_i. \tag{3.2} \]

Next, we note that $D - |A|$ and $D - |B|$ are elements of $\mathbb{Z}^{n,n}$ and, since $B[x]$ is irreducible, so is $(D - |B|)[x]$. Thus, from (2.20), we choose $x \in \mathbb{R}^n_+$ so that
\[ d_k - \sum_{j \in \pi} |b_{k,j}| x_j / x_k = l((D - |B|)[x]), \quad \text{for all} \quad k \in \pi. \]

With this choice of $x \in \mathbb{R}^n_+$, (3.2) implies that
\[ l((D - |B|)[x]) \geq \min \{ d_i - \sum_{j \in \pi} |a_{i,j}| y_j / y_i \}, \quad \text{for all} \quad y \in \mathbb{R}^n_+. \]

Then, the characterization of (2.19) directly yields
\[ l((D - |B|)[x]) \geq l((D - |A|)[x]). \tag{3.3} \]

If $B[x]$ is reducible, the argument used in (2.22) of Proposition 3 can be repeated to show that (3.3) again holds, whence $D - |B| \succeq D - |A|$. Conversely, if $D - |B| \succeq D - |A|$ for any real $D = \text{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n,n}$, assumed that $\pi$ is any nonempty subset of $\langle n \rangle$ with $B[x]$ irreducible, so that $(D - |B|)[x]$ is irreducible. By hypotheses, $l((D - |B|)[x]) \geq l((D - |A|)[x])$ for any real $D = \text{diag}[d_1, d_2, \ldots, d_n]$. For any $x \in \mathbb{R}^n_+$, define the real numbers
\[ d_i' = \begin{cases} \sum_{j \in \pi} |b_{i,j}| x_j / x_i, & \text{for all } i \in \pi, \\ 0, & \text{for all } i \notin \pi, \end{cases} \tag{3.4} \]

and set $\bar{D} = \text{diag}[d_1', d_2', \ldots, d_n'] \in \mathbb{R}^{n,n}$. With the definition of (3.4) and the inclusions of (2.18), it follows that
\[ d_i - \sum_{j \in \pi} |b_{i,j}| x_j / x_i = 0 = l((\bar{D} - |B|)[x]), \quad \text{for all } i \in \pi. \tag{3.5} \]

On the other hand, by hypothesis and by (2.18),
\[ l((\bar{D} - |B|)[x]) \geq l((\bar{D} - |A|)[x]) \geq \min \{ d_i - \sum_{j \in \pi} |a_{i,j}| y_j / y_i \}, \quad \text{for any} \quad y \in \mathbb{R}^n_+. \tag{3.6} \]

Thus, combining the results of (3.5)-(3.6), there is an $i \in \pi$ for which
\[ d_i - \sum_{j \in \pi} |b_{i,j}| x_j / x_i \geq \bar{d}_i - \sum_{j \in \pi} |a_{i,j}| y_j / y_i, \]

which implies that, for any $x$ and $y$ in $\mathbb{R}^n_+$, there is an $i \in \pi$ such that
\[ \sum_{j \in \pi} |a_{i,j}| y_j / y_i \geq \sum_{j \in \pi} |b_{i,j}| x_j / x_i, \]
i.e., $|A|$ dominates $|B|$.

Given $A$ and $B$ in $\mathbb{R}^{n,n}$, if there is a particular real diagonal matrix $D \in \mathbb{R}^{n,n}$ such that $D - |B| \succeq D - |A|$, then Proposition 5 provides no information as to whether or not $|A|$ dominates $|B|$. In fact, it is possible to construct an
example where $D - |B| \gg_r D - |A|$ for a particular real diagonal matrix, but where $|A|$ fails to dominate $|B|$. To show this, letting

$$|A| = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad |B| = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \\ 4 & \frac{1}{2} & 0 \end{bmatrix}; \quad |D| = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

it can be readily verified that $D - |B| \gg_r D - |A|$. On the other hand, it can be verified that $\rho(|A|) = \sqrt{2} \approx 1.414213$ and that $\rho(|B|) \approx 1.434467$, so that $\rho(|B|) > \rho(|A|)$. This, however, implies from Proposition 7 in the next section that $|A|$ cannot dominate $|B|$.

This brings us to our main result.

**Theorem 6** Given $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $C^{n,n}$, the following are equivalent:

i) $G((D + B)[z]) \subseteq G((D + A)[z])$ for any $\phi \neq \alpha \subseteq \langle n \rangle$ and for any $D = \text{diag}[d_1, d_2, \ldots, d_n] \in C^{n,n}$;

ii) $G((D + B)[z]) \subseteq G((D + A)[z])$ for any $\phi \neq \alpha \subseteq \langle n \rangle$ and for any $D = \text{diag}[d_1, d_2, \ldots, d_n] \in C^{n,n}$ such that $(d_i + b_{i,i})$ is real for all $i \in \langle n \rangle$;

iii) $|A|$ dominates $|B|$ and $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$;

iv) $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$ and $D - |B| \gg_r D - |A|$ for any real $D = \text{diag}[d_1, d_2, \ldots, d_n] \in R^{n,n}$.

**Proof** That (i) implies (ii) is obvious. Assuming (ii), we first observe, on choosing $z = \{i\}$ for any $i \in \langle n \rangle$, that (ii) implies from (1.1) that $d_i + a_{i,i} = d_i + b_{i,i}$ for any $d_i$ such that $d_i + b_{i,i}$ is real, whence $a_{i,i} = b_{i,i}$ for all $i \in \{i\}$. Next, consider any $z$ with $\phi \neq \alpha \subseteq \langle n \rangle$ for which $B[z]$ is irreducible, and any $x \in R^n$. Now, define the numbers $\tilde{d}_i$ so that

$$\tilde{d}_i + b_{i,i} = \begin{cases} \sum_{j \in \alpha} b_{i,j} x_j x_i, & i \in \alpha, \\ 0, & i \notin \alpha, \end{cases}$$

which implies that $\tilde{D} := \text{diag}[\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n]$ satisfies the hypothesis of (ii).

With this choice of the $\tilde{d}_i$'s, it follows that

$$|z| - (\tilde{d}_i + b_{i,i}) = \sum_{j \in \alpha} b_{i,j} x_j x_i, \quad \text{for all } i \in \alpha, \quad (3.8)$$

is evidently satisfied for $z = 0$. Using the inclusions of (2.9), we see that (3.8) with $z = 0$ implies that $v(0; (\tilde{D} + B)[x]) = 0$, so that, from Proposition 1, $0 \in G((\tilde{D} + B)[x])$. Thus, hypothesis (i) implies $0 \in G((\tilde{D} + A)[x])$, whence from Proposition 1, $v(0; (\tilde{D} + A)[x]) \geq 0$. With the second inequality of (2.9), there is, for any $y \in R^n$, an $i \in \alpha$ for which

$$0 \leq v(0; (\tilde{D} + A)[x]) \leq |d_i + a_{i,i}| + \sum_{j \notin \alpha} |a_{i,j}| y_j y_i.$$
But as $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$, the above inequality becomes, with the definition of (3.7), just
\[ \sum_{j \neq i} |b_{i,j}| x_j / x_i \leq \sum_{j \neq i} |a_{i,j}| y_j / y_i, \]
\[ i.e., |A| \text{ dominates } |B|, \text{ and (ii) implies (iii). That (iii) implies (iv) follows directly from Proposition 5.} \]

Finally, we show that (iv) implies (i). Assuming (iv), consider any $z$ with $\phi \neq z \leq \langle n \rangle$ and any $D = \text{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{C}^{n \times n}$. To establish (i), it suffices to show that if $z \in \partial G((D + B)[z])$, then $z \in G((D + A)[z])$. Moreover, because of (2.2), we may assume that $(D + B)[z]$ is irreducible. Thus, as $z \in \partial G((D + B)[z])$, it follows from Proposition 2 that there is an $x \in \mathbb{R}^n_+$ for which (cf. (2.11))
\[ |z - (d_i + b_{i,i})| = \sum_{j \neq i} |b_{i,j}| x_j / x_i, \quad \text{for all } i \in \langle n \rangle. \] (3.9)

Next, with the definition of the real numbers $d_i$ in (3.4), we see that the associated matrix $D' = \text{diag}[d_1, d_2, \ldots, d_n]$ is real, and the hypothesis of (iv) gives us that $D' - |B| \geq D' - |A|$. Now, from Proposition 3, this implies that, for any $y \in \mathbb{R}^n_+$, there exist integers $i$ and $k$ in $z$ such that
\[ d_i - \sum_{j \neq i} |b_{i,j}| x_j / x_i \geq d_k - \sum_{j \neq k} |a_{k,j}| y_j / y_k. \]
Hence, from the definition of the $d_i$'s in (3.4) and the hypothesis that $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$, the above inequality reduces to
\[ \sum_{j \neq k} |a_{k,j}| y_j / y_k \geq \sum_{j \neq k} |b_{k,j}| x_j / x_k. \]
Coupling this with (3.9) and again using the fact that $a_{k,k} = b_{k,k}$ then yield
\[ \max \left\{ \sum_{j \neq k} |a_{k,j}| y_j / y_k - |z - (d_k + a_{k,k})| \right\} \geq 0. \]
Thus, from (2.10), we deduce that $v(z; (D + A)[z]) \geq 0$, whence, from Proposition 1, $z \in G((D + A)[z])$, i.e., (iv) implies (i). \[ \blacksquare \]

4. PROPERTIES ASSOCIATED WITH THE RELATION 'A' DOMINATES 'B'

We begin with any easy consequence of Definition 4. For notation, let $\rho(C)$ denote the spectral radius of any $C \in \mathbb{C}^{n \times n}$, i.e., $\rho(C) = \max \{ |\lambda| : \lambda \in \text{spec}(C) \}$.

**Proposition 7** Given $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{C}^{n \times n}$, then $|A|$ dominates $|B|$ implies that
\[ \rho((|A| + x)[z]) \geq \rho((|B| + x)[z]) \quad \text{for any } \phi \neq z \leq \langle n \rangle. \] (4.1)
Proof. If $|A|$ dominates $|B|$, then, from Proposition 5 with $D = 0$, we have that $-|B| \geq -|A|$, or equivalently, that
\[
I(-(B)[x]) \geq I(-(A)[x]) \quad \text{for any} \quad \phi \neq \alpha \leq \langle n \rangle.
\] (4.2)
But, from the Perron–Frobenius theory of nonnegative matrices, it is easy to see from (2.19) that $I(-(B)[x]) = -\rho((B)[x])$, and similarly $I(-(A)[x]) = \rho((A)[x])$, whence (4.2) implies (4.1). \qed

For the next result, we write in the usual notation of nonnegative matrices that $|A| \geq |B|$ if $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{C}^{n \times n}$ satisfy
\[
|a_{i,j}| \geq |b_{i,j}| \quad \text{for all} \quad i, j \in \langle n \rangle.
\] (4.3)

**Proposition 8.** Given $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{C}^{n \times n}$ satisfying (1.3), then $|A| \geq |B|$ implies that $|A|$ dominates $|B|$.

Proof. If $|A| \geq |B|$, then for any nonempty $\alpha \leq \langle n \rangle$, for any real $D = \text{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n \times n}$, and for any $x \in \mathbb{R}^n$,
\[
d_i - \sum_{j \in \alpha} |b_{i,j}| x_j x_j \geq d_i - \sum_{j \in \alpha} |a_{i,j}| x_j x_j \quad \text{for all} \quad i \in \alpha.
\] (4.4)

Hence, since $D - |A|$ and $D - |B|$ are elements of $\mathbb{Z}^{n \times n}$, it follows from (4.4) and the characterization of (2.19) that
\[
I((D - |B|)[x]) \geq I((D - |A|)[x]) \quad \text{for all nonempty} \quad \alpha \leq \langle n \rangle,
\]
whence $D - |B| \geq D - |A|$ for all real $D = \text{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n \times n}$.

Thus, applying Propositions 5, $|A|$ dominates $|B|$. \qed

Finally, we remark that the converse of Propositions 7 is false.

As a counterexample, consider
\[
|A| = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad |B| = \begin{bmatrix} 0 & 0.08 & \frac{4}{7} \\ 0 & 0 & 2 \\ 4 & \frac{4}{7} & 0 \end{bmatrix}.
\]

By direct computation, it can be verified that (4.1) is valid in this case for all $\phi \neq \alpha \leq \langle 3 \rangle$. However, $|A|$ does not dominate $|B|$. To see this, choose $\alpha = \langle 3 \rangle$, and $\hat{x} = [0.30, 1, 1]^T$ and $\hat{y} = [0.9091, 0.5001, 1]^T$ in $\mathbb{R}^3$. Then, as can be verified, the inequality of (3.1) of Definition 4 fails for every $i \in \langle 3 \rangle$. In the same vein, the converse of Proposition 8 is easily seen to be false, since, with
\[
|A| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad |B| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]
$|A|$ dominates $|B|$ and $|B|$ dominates $|A|$, but $|A| \neq |B|$.

**References**


