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$\mathcal{A}\mathcal{L}\mathcal{P}\mathcal{S}$: Matrices with Nonpositive Off-Diagonal Entries

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ABSTRACT

The purpose of this paper is to characterize and interrelate various degrees of stability and semipositivity for real square matrices having nonpositive off-diagonal entries. The major classes considered are the sets of diagonally stable, stable, and semipositive matrices, denoted respectively by $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{S}$. The conditions defining these classes are weakened, and the resulting classes are examined. Their relationship to the classes of real matrices $\mathcal{P}$ and $\mathcal{P}_0$, whose off-diagonal entries are nonpositive and whose principal minors are respectively all positive and all nonnegative, is also included.

1. INTRODUCTION

In a previous paper [1], classes of real matrices, resulting from various degrees of stability and semipositivity, have been examined and interrelated. The four major classes considered there are the classes of diagonally stable

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matrices, stable matrices, matrices whose principal minors are all positive, and semipositive matrices, denoted by $\mathcal{Q}$, $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{S}$, respectively. The purpose of this paper is to examine and interrelate these four classes and their variants, with the restriction that all real matrices considered have only nonpositive off-diagonal entries; i.e., using the notation introduced by Fiedler and Pták [3], the matrices considered are in $\mathbf{Z}^{n,n}$ [cf. (2.1)]. As is well known, elements of $\mathbf{Z}^{n,n}$ appear frequently in many applications of the physical sciences (cf. Plemmons [5]).

In Sec. 2, we give the necessary notations and preliminaries; one of our main results appears in Sec. 3, and further extensions appear in Sec. 4.

2. NOTATION AND PRELIMINARIES

For n a positive integer, $\mathbf{R}^{n,n}$ denotes as usual the collection of all real $n\times n$ matrices $A=[a_{i,j}]$. Then, following Fiedler and Pták [3], the subset $\mathbf{Z}^{n,n}$ of $\mathbf{R}^{n,n}$ is defined as

$$\mathbf{Z}^{n,n}:=\{A=[a_{i,j}]\in\mathbf{R}^{n,n}: a_{i,j}\leqslant 0 \text{ for all } i\neq j\}. \tag{2.1}$$

Next, for $A=[a_{i,j}]\in\mathbf{R}^{n,n}$, we write $A\geqslant\mathcal{O}$ if $a_{i,j}\geqslant 0$ for all i and j. If $\mathbf{R}^{n}$ denotes the collection of all real column n-vectors $\mathbf{x}=[x_1,x_2,\dots,x_n]^T$, then $\mathbf{x}>\mathbf{0}$ ($\mathbf{x}\geqslant\mathbf{0}$) implies that $x_i>0$ ($x_i\geqslant 0$) for all $1\leqslant i\leqslant n$.

As is well known, if $A\in\mathbf{R}^{n,n}$ is reducible, there is a permutation matrix $P\in\mathbf{R}^{n,n}$ for which PAP^T is in reduced normal form (cf. [8, p. 46]):

$$PAP^T=\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,k}\\ & A_{2,2} & \cdots & A_{2,k}\\ & & \ddots & \vdots\\ \bigcirc & & & A_{k,k}\end{bmatrix}, \tag{2.2i}$$

where each $A_{i,i}$, $1\leqslant i\leqslant k$, is either square and irreducible or a 1×1 null matrix. For our purposes here, it is convenient to define a 1×1 null matrix to be irreducible. Continuing, if A is irreducible, we write simply

$$A=[A_{1,1}]. \tag{2.2ii}$$

Next, as is well known, any $A\in\mathbf{Z}^{n,n}$ can be expressed as

$$A=\alpha I-B, \tag{2.3}$$

where $B \in \mathbf{R}^{n,n}$ satisfies $B \geqslant \mathcal{O}$, and α is a real scalar. If $\alpha \geqslant \rho(B)$, where $\rho(B)$ denotes the spectral radius of B, then A is an *M-matrix*. The collection of all M-matrices in $\mathbf{Z}^{n,n}$ is denoted by $\mathcal{P}_0$, while the subset of all nonsingular M-matrices is denoted by $\mathcal{P}$. We note that if $A \in \mathbf{Z}^{n,n}$ has the reduced form (2.2i), then $A \in \mathcal{P}_0$ iff $A_{i,i} \in \mathcal{P}_0$ for all i.

As analogously defined in [1], $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{S}$ denote the following subsets of $\mathbf{Z}^{n,n}$:

$$\mathcal{L} := \{A \in \mathbf{Z}^{n,n} : \exists X \in \mathbf{R}^{n,n} \text{ with } X \text{ symmetric and positive definite such that } AX + XA^T \text{ is positive definite}\}; \tag{2.4}$$

$$\mathcal{A} := \{A \in \mathbf{Z}^{n,n} : \exists \text{ positive diagonal matrix } D \in \mathbf{R}^{n,n} \text{ such that } AD + DA^T \text{ is positive definite}\}; \tag{2.5}$$

$$\mathcal{S} := \{A \in \mathbf{Z}^{n,n} : \exists \mathbf{x} \in \mathbf{R}^n \text{ with } \mathbf{x} > \mathbf{0} \text{ such that } A\mathbf{x} > \mathbf{0}\}. \tag{2.6}$$

These subsets of $\mathbf{Z}^{n,n}$ can be enlarged by weakening their respective hypotheses, giving rise to the sets

$$\mathcal{W}\mathcal{L} := \{A \in \mathbf{Z}^{n,n} : \exists X \in \mathbf{R}^{n,n} \text{ with } X \text{ symmetric and positive definite such that } AX + XA^T \text{ is positive semidefinite}\}; \tag{2.4$'$}$$

$$\mathcal{W}\mathcal{A} := \{A \in \mathbf{Z}^{n,n} : \exists \text{ positive diagonal matrix } D \in \mathbf{R}^{n,n} \text{ such that } AD + DA^T \text{ is positive semidefinite}\}; \tag{2.5$'$}$$

$$\mathcal{W}\mathcal{S} := \{A \in \mathbf{Z}^{n,n} : \exists \mathbf{x} \in \mathbf{R}^n \text{ with } \mathbf{x} > \mathbf{0} \text{ such that } A\mathbf{x} \geqslant \mathbf{0}\}. \tag{2.6$'$}$$

Next, if T is an arbitrary subset of $\mathbf{Z}^{n,n}$, then

$$\tilde{T} := \{A \in T : A \text{ is symmetric}\} \tag{2.7}$$

denotes the collection of all symmetric matrices in T. This then allows us to consider the subsets $\widetilde{\mathcal{W}\mathcal{A}}$, $\widetilde{\mathcal{W}\mathcal{L}}$, $\widetilde{\mathcal{W}\mathcal{S}}$, and $\tilde{\mathcal{P}}_0$ of $\mathbf{Z}^{n,n}$.

As subsets of $\mathbf{Z}^{n,n}$, it is well known (cf. Plemmons [5] and [9], and references included there) that

$$\mathcal{A} = \mathcal{L} = \mathcal{P} = \mathcal{S}, \tag{2.8}$$

and a continuity argument easily shows that $\mathcal{WL}$, $\mathcal{WA}$, and $\mathcal{WS}$ are all subsets of $\mathcal{P}_0$, i.e.,

$$\mathcal{WL} \subseteq \mathcal{P}_0, \qquad \mathcal{WA} \subseteq \mathcal{P}_0, \qquad \mathcal{WS} \subseteq \mathcal{P}_0. \tag{2.9}$$

The precise relationship among these last four sets is one of the main results (Theorem 1) of this paper, to be given in the next section.

3. FIRST MAIN RESULT

We begin by establishing

LEMMA 1. *Let $A \in \mathcal{P}_0$ be irreducible. Then, A is an element of $\mathcal{WA}$, $\mathcal{WS}$, and $\mathcal{WL}$.*

Proof. If $A \in \mathcal{P}$, i.e., if A is a nonsingular M-matrix, then from (2.8), A is in $\mathcal{A}$, $\mathcal{S}$, and $\mathcal{L}$, whence A is in $\mathcal{WA}$, $\mathcal{WS}$, and $\mathcal{WL}$ as well. Thus, we may assume that $A = [a_{i,j}] \in \mathbf{Z}^{n,n}$ is a singular irreducible M-matrix. Now, from (2.3), we can express A as $A = \rho(B)I - B$, where B is irreducible and satisfies $B \geqslant \mathcal{O}$. From the Perron-Frobenius theorem (cf. [8, p. 30]), let $\mathbf{x} > \mathbf{0}$ and $\mathbf{y} > \mathbf{0}$ in $\mathbf{R}^n$ be such that $B\mathbf{x} = \rho(B)\mathbf{x}$ and $B^T\mathbf{y} = \rho(B)\mathbf{y}$, so that $A\mathbf{x} = A^T\mathbf{y} = \mathbf{0}$. Thus [cf. (2.6′)], $A \in \mathcal{WS}$. In the spirit of Tartar's proof [7], define

$$d_i := x_i / y_i; \qquad c_{i,j} := y_i a_{i,j} x_j; \qquad 1 \leqslant i,j \leqslant n, \tag{3.1}$$

and set $D := \operatorname{diag}[d_1, d_2, \ldots, d_n]$, and $C := [c_{i,j}] \in \mathbf{Z}^{n,n}$. Note that D is a positive diagonal matrix and that C is irreducible with zero row and column sums. By definition, the real symmetric matrix $(C + C^T) \in \mathbf{Z}^{n,n}$ is then diagonally dominant with positive diagonal entries, and thus (cf. [8, p. 24, Exercise 4]) it is a positive semidefinite matrix:

$$\mathbf{w}^T(C + C^T)\mathbf{w} = 2\sum_{i,j=1}^{n} w_i(c_{i,j})w_j \geqslant 0 \qquad \forall \mathbf{w} \in \mathbf{R}^n. \tag{3.2}$$

But then, from (3.1) and (3.2), we have that

$$\mathbf{u}^T A D \mathbf{u} = \sum_{i,j=1}^{n} u_i a_{i,j} d_j u_j = \sum_{i,j=1}^{n} \left(\frac{u_i}{y_i}\right) c_{i,j} \left(\frac{u_j}{y_j}\right) \geqslant 0 \qquad \forall \mathbf{u} \in \mathbf{R}^n, \tag{3.3}$$

which is equivalent to $AD+DA^T$ being positive semidefinite. By definition [cf. (2.5′)], $A\in\mathcal{WA}$, and automatically [cf. (2.4′)], $A\in\mathcal{WL}$. ∎

Lemma 1, coupled with (2.8) and (2.9), shows that the differences between the various sets $\mathcal{WA}$, $\mathcal{WL}$, $\mathcal{WS}$, and $\mathcal{P}_0$ can occur only in the case of reducible singular matrices in $\mathcal{P}_0$.

One of our main results is

THEOREM 1. *Let $A\in\mathcal{P}_0$. The, with the notation of* (2.2),

$$\left.\begin{array}{l}\text{(i) } A\in\mathcal{WA} \textit{ iff } A_{i,i} \textit{ singular implies } A_{i,j}=A_{j,i}=\mathcal{O} \textit{ for all } j\neq i;\\ \text{(ii) } A\in\mathcal{WS} \textit{ iff } A_{i,i} \textit{ singular implies } A_{i,j}=\mathcal{O} \textit{ for all } j\neq i;\\ \text{(iii) } A\in\mathcal{WL} \textit{ iff } A_{i,i} \textit{ and } A_{j,j} \textit{ singular, } j>i, \textit{ imply } A_{i,j}=\mathcal{O}.\end{array}\right\} \tag{3.4}$$

Consequently, as subsets of $\mathbf{Z}^{n,n}$,

$$\mathcal{WA}\subseteq\mathcal{WS}\subseteq\mathcal{WL}\subseteq\mathcal{P}_0. \tag{3.5}$$

Moreover, while equality trivially holds in (3.5) *for* $n=1$, *strict inclusion holds throughout* (3.5) *for every* $n\geqslant 2$. *Finally, in the symmetric case* [*cf.* (2.7)], *for every* $n\geqslant 1$,

$$\widetilde{\mathcal{WA}}=\widetilde{\mathcal{WS}}=\widetilde{\mathcal{WL}}=\tilde{\mathcal{P}}_0. \tag{3.6}$$

Proof. Note that if $A\in\mathcal{P}_0$ is irreducible, conditions (i)–(iii) all vacuously hold, implying that A is in $\mathcal{WA}$, $\mathcal{WS}$, and $\mathcal{WL}$, as established in Lemma 1. Thus, we may suppose that A is reducible, and is in the form (2.2i).

(i) Suppose that each $A_{i,i}$ singular in (2.2i) implies that $A_{i,j}=A_{j,i}=\mathcal{O}$ for all $j\neq i$ in (2.2i). Then any singular $A_{i,i}$ is, in the sense of directed graphs, disjoint from any other of the $A_{j,j}$'s, so that, by a suitable permutation of the blocks of (2.2i), we have

$$\hat{P}A\hat{P}^T=\begin{bmatrix}\hat{A}_{1,1} & & & & \bigcirc\\ & \hat{A}_{2,2} & & & \\ & & \ddots & & \\ & & & \hat{A}_{r,r} & \\ \bigcirc & & & & \hat{B}_{r+1,r+1}\end{bmatrix}, \tag{3.7}$$

where $\hat{A}_{i,i}$ is a singular irreducible M-matrix for $1\leqslant i\leqslant r$, and where $\hat{B}_{r+1,r+1}$

is an upper triangular nonsingular M-matrix. Thus, $\hat{B}_{r+1,r+1}$, an element of $\mathcal{P}$, is from (2.8) in $\mathcal{Q}$ and hence in $\mathcal{W}\mathcal{Q}$, while from Lemma 1, each $\hat{A}_{i,i}$ is in $\mathcal{W}\mathcal{Q}$, $1 \leqslant i \leqslant r$. Clearly, $\hat{P}A\hat{P}^T$, the direct sum of matrices in $\mathcal{W}\mathcal{Q}$, is then in $\mathcal{W}\mathcal{Q}$, whence $A \in \mathcal{W}\mathcal{Q}$.

Conversely, let $A \in \mathcal{W}\mathcal{Q}$ have the form (2.2i), and suppose, on the contrary, that a singular irreducible $A_{i,i}$ can be imbedded into a larger principal submatrix A_s of A of the form

$$A_s = \begin{bmatrix} A_{i,i} & \mathbf{x} \\ 0 & a \end{bmatrix}, \tag{3.8}$$

where a is a scalar, and where $\mathbf{x} \neq \mathbf{0}$. (Because A is an element of $\mathbf{Z}^{n,n}$, note that $\mathbf{x} \leqslant \mathbf{0}$.) Now, since $A \in \mathcal{W}\mathcal{Q}$ by hypothesis, it easily follows that $A_s \in \mathcal{W}\mathcal{Q}$. Hence there exists a positive diagonal matrix D_s with

$$D_s = \begin{bmatrix} D & 0 \\ 0 & d \end{bmatrix}, \tag{3.9}$$

where d is a positive scalar, such that [cf. (3.3)]

$$\mathbf{u}_s^T A_s D_s \mathbf{u}_s \geqslant 0 \qquad \forall \mathbf{u}_s. \tag{3.10}$$

Next, because $A_{i,i}$ is irreducible and singular, let $\mathbf{u} > \mathbf{0}$ be such that $A_{i,i}D\mathbf{u} = \mathbf{0}$, and consider $\mathbf{u}_s := [\mathbf{u}, \epsilon]^T$, where ϵ is a scalar. Then, we directly find from (3.8) and (3.9) that

$$\mathbf{u}_s^T A_s D_s \mathbf{u}_s = \epsilon d\,(\mathbf{u}^T\mathbf{x} + \epsilon a).$$

But, because $\mathbf{x} \leqslant \mathbf{0}$ with $\mathbf{x} \neq \mathbf{0}$, the above is negative for all $\epsilon > 0$ sufficiently small, which contradicts (3.10). Thus, $\mathbf{x} = \mathbf{0}$. A similar argument applied to

$$\begin{bmatrix} a & y^T \\ 0 & A_{i,i} \end{bmatrix}$$

shows that $\mathbf{y}^T = \mathbf{0}$, completing the proof of (3.4i).

(ii) Suppose that each $A_{i,i}$ which is singular in (2.2i) implies that $A_{i,j} = \mathcal{O}$ for all $j \neq i$. If $A_{k,k}$ in (2.2i) is singular, then, by construction, $A_{k,k}$ is a singular irreducible M-matrix, and thus there is a $\mathbf{y}_k > \mathbf{0}$ for which $A_{k,k}\mathbf{y}_k = \mathbf{0}$. Similarly, if $A_{k,k}$ in (2.2i) is nonsingular, then $A_{k,k}$ is a nonsingular irreducible M-matrix, and there is a $\mathbf{y}_k > \mathbf{0}$ for which $A_{k,k}\mathbf{y}_k > \mathbf{0}$. Recalling that $A_{i,j} \leqslant \mathcal{O}$ for all $i \neq j$, and that $A_{i,i}$ singular by hypothesis implies $A_{i,j} = \mathcal{O}$ for all $j \neq i$, it can be verified that the $\mathbf{y}_i$'s can be scaled so that $\mathbf{x} := [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k]^T$ satisfies $\mathbf{x} > \mathbf{0}$ and

$$\sum_{j \geqslant i} A_{i,j}\mathbf{y}_j \geqslant \mathbf{0} \qquad \text{for all} \quad 1 \leqslant i \leqslant k,$$

whence $PAP^T\mathbf{x} \geqslant \mathbf{0}$. Hence, $A \in \mathcal{WS}$.

Conversely, assume that $A \in \mathcal{WS}$, and let $\mathbf{x} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k]^T > \mathbf{0}$ be such that $PAP^T\mathbf{x} \geqslant \mathbf{0}$, i.e., from (2.2i),

$$\sum_{j \geqslant i} A_{i,j}\mathbf{y}_j \geqslant \mathbf{0} \qquad \text{for all} \quad 1 \leqslant i \leqslant k. \tag{3.11}$$

Because $A_{i,j} \leqslant \mathcal{O}$ for all $j > i$, it follows that $A_{i,i}\mathbf{y}_i \geqslant \mathbf{0}$ for all $1 \leqslant i \leqslant k$. Now, suppose that $A_{i,i} \in \mathbf{Z}^{r,r}$ is a singular irreducible M-matrix. Using the well-known min-max formulation (cf. [8, p. 32]) for the spectral radius of a nonnegative irreducible matrix, it follows from the representation (2.3) that for any $\mathbf{w} > \mathbf{0}$ in $\mathbf{R}^r$,

$$\min_j \left\{ \frac{(A_{i,i}\mathbf{w})_j}{w_j} \right\} < 0 \quad \text{or} \quad \left\{ \frac{(A_{i,i}\mathbf{w})_j}{w_j} \right\} = 0 \quad \text{for all } j, \tag{3.12}$$

since $A_{i,i}$ is a singular irreducible M-matrix. However, from the above we have that $A_{i,i}\mathbf{y}_i \geqslant \mathbf{0}$, which implies that

$$\min_j \left[\frac{(A_{i,i}\mathbf{y}_i)_j}{(\mathbf{y}_i)_j} \right] \geqslant 0.$$

Thus, from (3.12), $A_{i,i}\mathbf{y}_i = \mathbf{0}$, and hence, because $A_{i,j} \leqslant \mathcal{O}$ for all $j > i$, it further follows from (3.11) that $A_{i,j}\mathbf{y}_j = \mathbf{0}$ for all $j > i$. Hence, since $\mathbf{y}_j > \mathbf{0}$, $A_{i,j} = \mathcal{O}$ for all $j > i$, which completes the proof of (3.4ii).

Continuing, (3.4iii) is a direct consequence of Schneider [6] and Carlson and Schneider [2], which completes the proof of (3.4). The inclusions of (3.5) are a direct consequence.

Omitting the trivial case $n = 1$, we now show that the inclusions of (3.5) are strict for every $n \geqslant 2$. Consider the following matrices in $\mathbf{Z}^{n,n}$, $n \geqslant 2$:

$$A_1 := \left[\begin{array}{cc|c} 1 & -1 & \multirow{2}{*}{$\bigcirc$} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{\bigcirc} & I_{n-2} \end{array}\right], \qquad A_2 := \left[\begin{array}{cc|c} 0 & -1 & \multirow{2}{*}{$\bigcirc$} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{\bigcirc} & I_{n-2} \end{array}\right],$$

$$A_3 := \left[\begin{array}{cc|c} 0 & -1 & \multirow{2}{*}{$\bigcirc$} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{\bigcirc} & I_{n-2} \end{array}\right],$$

where the identity matrix, $I_{n-2} \in \mathbf{R}^{n-2,n-2}$, is to be deleted in the above matrices if $n=2$. It is easily verified from (i)–(iii) that $A_1 \notin \mathcal{WA}$, $A_1 \in \mathcal{WS}$; $A_2 \notin \mathcal{WS}$, $A_2 \in \mathcal{WL}$; $A_3 \notin \mathcal{WL}$, $A_3 \in \mathcal{P}_0$.

Finally, in the symmetric case, (i)–(iii) again directly give (3.6), since (2.2i) reduces to the direct sum of the $A_{i,i}$'s when A is symmetric. ■

4. EXTENSIONS

There are a variety of ways, as considered in [1], in which the sets $\mathcal{A}$, $\mathcal{P}$, $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{WA}$, $\mathcal{P}_0$, $\mathcal{WL}$, $\mathcal{WS}$, considered specifically as subsets of $\mathbf{Z}^{n,n}$, can be either restricted or enlarged, and the relationship between these sets can then be pursued, in the spirit of extending the inclusions of (2.8) and (3.5). In this section, for brevity, just one such extension will be considered.

In analogy with (2.4)–(2.6) and (2.4′)–(2.6′), we define

$$\begin{aligned} \mathcal{VWL} = \{ A \in \mathbf{Z}^{n,n} : \exists X \in \mathbf{R}^{n,n} &\text{ with } X \text{ symmetric, } X \neq \mathcal{O}, \text{ and} \\ &X \text{ positive semidefinite such that } AX + XA^T \text{ is} \\ &\text{positive semidefinite}\}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \mathcal{VWA} := \{ A \in \mathbf{Z}^{n,n} : \exists &\text{ nonnegative diagonal matrix } D \in \mathbf{R}^{n,n} \\ &\text{with } D \neq \mathcal{O} \text{ such that } AD + DA^T \text{ is positive semidefinite}\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \mathcal{VWS} := \{ A \in \mathbf{Z}^{n,n} : \exists \mathbf{x} \in \mathbf{R}^n &\text{ with } \mathbf{x} \geqslant \mathbf{0} \text{ and } \mathbf{x} \neq \mathbf{0} \text{ such that} \\ &A\mathbf{x} \geqslant \mathbf{0}\}. \end{aligned} \tag{4.3}$$

LEMMA 2. *Let $A \in \mathbf{Z}^{n,n}$ be irreducible. Then $A \in \mathcal{V}\mathcal{W}\mathcal{S}$ iff $A \in \mathcal{P}_0$.*

Proof. Consider any irreducible $A \in \mathcal{V}\mathcal{W}\mathcal{S}$. From (4.3), there is an $\mathbf{x} \in \mathbf{R}^n$ with $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ for which $A\mathbf{x} \geqslant \mathbf{0}$. Suppose that $\mathbf{x}$ has some zero components, i.e., after a suitable permutation of indices, $\mathbf{x} = [\mathbf{y}_1, \mathbf{y}_2]^T$, where $\mathbf{y}_1 > \mathbf{0}$ and $\mathbf{y}_2 = \mathbf{0}$. Partition A conformally; then $A\mathbf{x} \geqslant \mathbf{0}$ implies that

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} A_{1,1}\mathbf{y}_1 \\ A_{2,1}\mathbf{y}_1 \end{bmatrix} \geqslant \mathbf{0}. \tag{4.4}$$

But, as $A_{2,1} \leqslant \mathcal{O}$ and $\mathbf{y}_1 > \mathbf{0}$, evidently $A_{2,1} = \mathcal{O}$. As this contradicts the hypothesis that A is irreducible, $\mathbf{x} > \mathbf{0}$. Hence, by definition, $A \in \mathcal{W}\mathcal{S}$. Thus, from (3.5) of Theorem 1, $A \in \mathcal{P}_0$. Conversely if $A \in \mathcal{P}_0$ is irreducible, then from Lemma 1, $A \in \mathcal{W}\mathcal{S}$, whence $A \in \mathcal{V}\mathcal{W}\mathcal{S}$. ■

LEMMA 3. *For any $n \geqslant 1$, $\mathcal{V}\mathcal{W}\mathcal{C} = \mathcal{V}\mathcal{W}\mathcal{S}$.*

Proof. Since it is known from [1, Proposition 15] that

$$\mathcal{V}\mathcal{W}\mathcal{C} \subseteq \mathcal{V}\mathcal{W}\mathcal{S} \tag{4.5}$$

as general subsets of $\mathbf{R}^{n,n}$, it suffices to show that the reverse inclusion holds in (4.5) in $\mathbf{Z}^{n,n}$. Note first from Lemma 2 that any irreducible $A \in \mathcal{V}\mathcal{W}\mathcal{S}$ is in $\mathcal{P}_0$, so that from Lemma 1, $A \in \mathcal{W}\mathcal{C}$, whence $A \in \mathcal{V}\mathcal{W}\mathcal{C}$. Thus, only the reducible case remains.

Suppose then that $A \in \mathcal{V}\mathcal{W}\mathcal{S}$ is reducible, and suppose that A has the reduced normal form of (2.2i), where the diagonal blocks $A_{i,i}$ are irreducible. Then there is an $\mathbf{x} \in \mathbf{R}^n$ with $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ for which $A\mathbf{x} \geqslant \mathbf{0}$. Partitioning $\mathbf{x}$ conformally with respect to the partitioning of (2.2), we can express $\mathbf{x}$ as $\mathbf{x} = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k]^T$, where $\mathbf{y}_j \geqslant \mathbf{0}$ for $1 \leqslant j \leqslant k$, and where at least one $\mathbf{y}_j \neq \mathbf{0}$. Moreover, $A\mathbf{x} \geqslant \mathbf{0}$ implies that

$$\sum_{j \geqslant i} A_{i,j}\mathbf{y}_j \geqslant \mathbf{0} \qquad \text{for all} \quad 1 \leqslant i \leqslant k. \tag{4.6}$$

Since $A_{i,j} \leqslant \mathcal{O}$ for all $j > i$ and since $\mathbf{y}_j \geqslant \mathbf{0}$, it follows from (4.6) that

$$A_{i,i}\mathbf{y}_i \geqslant \mathbf{0} \qquad \text{for all} \quad 1 \leqslant i \leqslant k. \tag{4.7}$$

Now, consider any $\mathbf{y}_j \neq \mathbf{0}$. Because $A_{j,j}$ is irreducible, it necessarily again follows [cf. (4.4)] that $\mathbf{y}_j > \mathbf{0}$. Thus, $\mathbf{y}_j \neq \mathbf{0}$ implies $\mathbf{y}_j > \mathbf{0}$, and $A_{j,j}$ is, from (4.7), necessarily in $\mathcal{WS}$, whence $A_{j,j} \in \mathcal{P}_0$ from (3.5). Now, define the nonempty set

$$\Omega := \{ j : 1 \leqslant j \leqslant k \text{ and } \mathbf{y}_j > \mathbf{0} \},$$

and let Ω' be its complement relative to $1 \leqslant i \leqslant k$. As a consequence of (4.6),

$$A_{i,j} = \mathcal{O} \qquad \forall j > i \quad \text{with} \quad i \in \Omega', \quad j \in \Omega. \tag{4.8}$$

Next, let r be the first positive integer in Ω. If $r > 1$, then (4.8) implies that $A_{i,r} = \mathcal{O}$ for all $i < r$. Thus, with the assumed triangular form of A in (2.2i), we have (whether $r > 1$ or $r = 1$) that

$$A_{i,r} = \mathcal{O} \qquad \text{for all} \quad i \neq r. \tag{4.9}$$

Now, define the block diagonal matrix $E := \operatorname{diag}[E_1, E_2, \ldots, E_k]$ where $E_r := I$, and $E_j := \mathcal{O}$ for all $j \neq r$, and where the partitioning for E is conformal with that of A in (2.2i). By construction, then, the product AE is a block diagonal matrix of the form

$$AE = \operatorname{diag}[\mathcal{O}, \ldots, \mathcal{O}, A_{r,r}, \mathcal{O}, \ldots, \mathcal{O}].$$

Now, applying (3.4i) of Theorem 1, we have $AE \in \mathcal{WC}$, which implies [cf. (4.2)] that $A \in \mathcal{VWC}$. ∎

With Lemmas 2 and 3, we have

Theorem 2. *Let* $A \in \mathbf{Z}^{n,n}$. *Then, with the notation of* (2.2),

(i) $A \in \mathcal{VWS}$ *iff there is an* r *such that* $A_{r,r} \in \mathcal{P}_0$, *with* $A_{i,r} = \mathcal{O}$ *for all* $i \neq r$;

(ii) $A \in \mathcal{VWL}$ *iff* A *has an eigenvalue* λ *with* $\operatorname{Re}\lambda \geqslant 0$.

As subsets of $\mathbf{Z}^{n,n}$, $\mathcal{VWC} = \mathcal{VWS}$, *and*

$$\mathcal{P}_0 \subseteq \mathcal{VWS} \subseteq \mathcal{VWL}. \tag{4.10}$$

Moreover, while equality trivially holds throughout in (4.10) *for* $n = 1$, *strict inclusion holds throughout* (4.10) *for every* $n \geqslant 2$. *Finally, in the symmetric* *case* [*cf.* (2.7)],

$$\tilde{\mathcal{P}}_0 \subseteq \widetilde{\mathcal{VWS}} \subseteq \widetilde{\mathcal{VWL}}, \tag{4.11}$$

where equality holds trivially in (4.11) *for* $n=1$, *while strict inclusion is valid throughout there for every* $n \geqslant 2$.

Proof. To establish (i), it suffices to show, as a consequence of Lemma 2 and the proof of Lemma 3 [cf. (4.9)], that if $A \in \mathbf{Z}^{n,n}$ is reducible with $A_{r,r} \in \mathcal{P}_0$ and with $A_{i,r} = \mathcal{O}$ for all $i \neq r$, then $A \in \mathcal{VWS}$. Writing $\mathbf{x} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k]^T$, we can choose $\mathbf{y}_r > \mathbf{0}$ such that $A_{r,r}\mathbf{y}_r \geqslant \mathbf{0}$, since $A_{r,r} \in \mathcal{P}_0$ is irreducible. Defining $\mathbf{y}_j := \mathbf{0}$ for all $j \neq r$, then it directly follows that $A\mathbf{x} \geqslant \mathbf{0}$, where $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$. Thus [cf. (4.3)], $A \in \mathcal{VWS}$, completing the proof of (i). That (ii) holds is a direct consequence of [1, Proposition 5].

The inclusion $\mathcal{P}_0 \subseteq \mathcal{VWS}$ now follows from (i), or from Fiedler and Pták [4] or [1, Theorem 4], so that from Lemma 3, $\mathcal{P}_0 \subseteq \mathcal{VWA}$. Next, from the definitions of (4.1) and (4.2), it is evident that $\mathcal{VWA} \subseteq \mathcal{VWL}$, which establishes (4.10).

Concerning the sharpness of the inclusions of (4.10), consider $A := \operatorname{diag}[-1, 0, \ldots, 0] \in \mathbf{R}^{n,n}$ for every $n \geqslant 2$. It is obvious from (i) that $A \in \mathcal{VWS}$, but $A \notin \mathcal{P}_0$. Next, for every $n \geqslant 2$, consider the upper bidiagonal matrix $A \in \mathbf{Z}^{n,n}$:

$$A = \begin{bmatrix} -1 & -1 & & \bigcirc \\ & \ddots & \ddots & \\ & & -1 & -1 \\ \bigcirc & & & 0 \end{bmatrix}.$$

Since A has an eigenvalue zero, $A \in \mathcal{VWL}$ from (ii). On the other hand, A fails to satisfy (i), whence $A \notin \mathcal{VWS}$.

Finally, in the symmetric case, (i)–(ii) directly give (4.11), as well as the statements concerning the inclusions of (4.11) for the cases $n=1$ and $n \geqslant 2$. ∎

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