ON INCOMPLETE POLYNOMIALS. II

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Reprinted from Pacific Journal of Mathematics Vol. 92, No. 1
1981
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The approximation of $x^n$ by incomplete polynomials is studied, i.e., we consider the extremal problem

$$E_{n-k,k} = \inf \left\{ \left\| x^n - \sum_{j=1}^{k} d_j x^{n-j} \right\|_{[0,1]} : (d_1, \ldots, d_k) \in R^k \right\}, \quad n \geq k,$$

for the supremum norm on $[0,1]$. We show that, for $k$ fixed, $n^k E_{n-k,k} \rightarrow \varepsilon_k$ as $n \rightarrow \infty$, where

$$\varepsilon_k = \inf \left\{ \left\| e^{-t} \left( t^k + \sum_{j=1}^{k-1} a_j t^j \right) \right\|_{[0,\infty)} : (a_0, \ldots, a_{k-1}) \in R^k \right\}.$$

A generalization of this result for the case of lacunary polynomial approximation is given, as well as inequalities for $E_{n-k,k}$ and $\varepsilon_k$. Furthermore, we prove that for any polynomial $P(t)$ of degree at most $k$, there holds for the supremum norm $\|e^{-t}P(t)\|_{[0,\infty)} = \|e^{-t}P(t)\|_{[0,1]}$.

1. Introduction. In this note, we continue our investigation [6], [7], [8], [3], of incomplete polynomials, a subject first introduced by G. G. Lorentz [4]. Following the notation of [7], if $\pi_n$ denotes the set of all real polynomials of degree at most $n$, then for each pair $(s,k)$ of nonnegative integers, $\pi_{s,k}$ denotes the set of polynomials

$$\pi_{s,k} = \{ x^s q_k(x) : q_k \in \pi_k \},$$

so that $\pi_{s,k} \subset \pi_{s+1,k}$. A polynomial in $\pi_{s,k}$ is called an incomplete polynomial of type $(s,k)$. For any set $K \subset R$, the norm $\| \cdot \|_K$ shall denote the supremum norm on $K$, i.e., $\| g \|_K = \sup \{ |g(x)| : x \in K \}$.

Setting

$$E_{s,k} = \inf \{ \| x^s (x^k - q(x)) \|_{[0,1]} : q \in \pi_{s,k} \}, \quad \pi_{-i} = [0],$$

it is known [7] that, for each pair $(s,k)$, there exists a unique monic polynomial $Q_{s,k}(x) \in \pi_{s,k}$ of exact degree $s + k$, such that $\| Q_{s,k} \|_{[0,1]} = E_{s,k}$.

In a recent paper, Borosh, Chui, and Smith [1] established that for any positive integer $k$, there exist positive constants $\sigma_i(k)$ and $\sigma_i(k)$ such that

$$\sigma_i(k) \leq n^k E_{n-k,k} \leq \sigma_i(k), \quad \forall n > k.$$

They also proved that the coefficients of the extremal polynomials $Q_{s-k,k}(x)$ are bounded as $n \rightarrow \infty$.

One aim of this note is to derive (cf. (3.3)) explicit upper and
lower bounds for $n^k E_{n-k,b}$ for all $n > k$. But more importantly, we prove (in Corollary 2.3) that the limit of $n^k E_{n-k,b}$ as $n \to \infty$ exists and is given precisely by

\begin{equation}
\lim_{n \to \infty} n^k E_{n-k,b} = \varepsilon_k, \quad \forall k \geq 0,
\end{equation}

where

\begin{equation}
\varepsilon_k := \inf\{|e^{tf(t)} - p(t)|_{[0, \infty]} : p \in \pi_{k-1}\}.
\end{equation}

Since, after suitable normalization, the extremal polynomials for (1.2) are $L^2$-analogos of Jacobi polynomials, and the extremal polynomials for (1.5) are $L^2$-versions of Laguerre polynomials, our essential contribution is to show, as in the $L^1$-setting, that $L^2$-Laguerre polynomials can be obtained as the limit of certain $L^2$-Jacobi polynomials.

A generalization of the inequalities (1.3) to the case of lacunary polynomials is also proved in [1]. As a consequence of our main theorem, we derive the best possible result for this case as well (cf. Corollary 2.2).

The outline of the paper is as follows. In §2, we prove our main result and deduce as corollaries the limit (1.4) and its analog for lacunary polynomials. In §3, we derive inequalities for incomplete polynomials, and in §4 we study the quantities $\varepsilon_k$ of (1.5). We also prove in §4 (cf. Theorem 4.2) that for any polynomial $P \in \pi_k$, there holds

\begin{equation}
\|e^{tf(t)} - P(t)\|_{[0, \infty)} = \|e^{tf(t)} - P(t)\|_{[0, 2k]}.
\end{equation}

Moreover, the interval $[0, 2k]$ is best possible in an asymptotic sense, as $k \to \infty$.

\section{2. The main result and its consequences.}

Our primary objective is to prove

\begin{theorem}
Let the $k+1$ integers $0 \leq \mu_1 < \mu_2 < \cdots < \mu_k < m$ be fixed, and, for each nonnegative integer $n$, set

\begin{equation}
E_n := \inf \left\{ x^n \left( x^m - \sum_{j=1}^k c_j x^{\mu_j} \right) \right\} : (c_1, \cdots, c_k) \in \mathbb{R}^k \right\}.
\end{equation}

Then,

\begin{equation}
\lim_{n \to \infty} n^k E_n = \varepsilon_k \prod_{j=1}^k (m - \mu_j),
\end{equation}

where $\varepsilon_k$ is defined in (1.5).
\end{theorem}
Before proceeding with the proof, we remark that by Descartes’ rule of signs, the linear space generated by the functions \( x^{s+m}, x^{s+2m}, \ldots, x^{s+nm} \) satisfies the Haar condition on \((0, 1)\), and hence (cf. [5, p. 16]), the extremal problem (2.1) has a unique solution for each \( n \geq 0 \), i.e., there exist unique constants \( c_j^n(n) \), \( j = 1, \ldots, k \), such that

\[
E_n = \left\| x^n \left( x^n - \sum_{j=1}^{k} c_j^n(n) x^{sj} \right) \right\|_{L^1[0,1]}, \quad \forall n \geq 0.
\]

For notational convenience, we set

\[
p^n_*(x) = x^n - \sum_{j=1}^{k} c_j^n(n) x^{sj}, \quad \forall n \geq 0.
\]

The proof of Theorem 2.1 requires the following lemmas:

**Lemma 1.** (Lorentz [4], Saff and Varga [6]). Let \( P(x) \neq 0 \) be an incomplete polynomial of type \((s, m)\), where \( s + m > 0 \). If \( |P(\zeta)| = ||P||_{L^1[0,1]} \) with \( \zeta \in [0, 1] \), then \( s^j/(s + m)^j \leq \zeta \leq 1 \).

**Lemma 2.** For each \( n \), the polynomial \( x^n p^n_*(x) \) (cf. (2.4)) has an alternation set of \( k + 1 \) points in \([n^j/(n + m)^j, 1]\), i.e., there exist points \( \{\zeta_{i,n}\}_{i=0}^k \) with

\[
n^j/(n + m)^j \leq \zeta_{0,n} < \cdots < \zeta_{k,n} \leq 1
\]

such that

\[
(\zeta_{i,n}) p^n_*(\zeta_{i,n}) = (-1)^{i-1} E_n, \quad i = 0, 1, \ldots, k.
\]

Consequently, \( p^n_*(x) \) has \( k \) distinct zeros in \( (n^j/(n + m)^j, 1) \), and, as \( n \to \infty \), these \( k \) zeros all tend to \( x = 1 \).

**Proof.** From the fundamental property of Haar system approximation (cf. [5, p. 20]), \( x^n p^n_*(x) \) has an alternation set consisting of \( k + 1 \) points in \([0, 1]\) and, by Lemma 1, these points must all belong to \([n^j/(n + m)^j, 1]\). \( \square \)

**Lemma 3.** Set \( \mu_{s+1} = m \). Then, for the polynomials \( p^n_*(x) \) of (2.4), we have, uniformly on each compact set of \( R \),

\[
\lim_{n \to \infty} p^n_*(x) = \frac{1}{A} \det \begin{bmatrix}
C(\mu_1, 0) & C(\mu_1, 0) & \cdots & C(\mu_{s+1}, 0) \\
C(\mu_1, 1) & C(\mu_1, 1) & \cdots & C(\mu_{s+1}, 1) \\
\vdots & \vdots & \ddots & \vdots \\
C(\mu_1, k-1) & C(\mu_1, k-1) & \cdots & C(\mu_{s+1}, k-1) \\
x^{\mu_1} & x^{\mu_2} & \cdots & x^{\mu_{s+1}}
\end{bmatrix},
\]

where \( A = C(\mu_1, 0) C(\mu_1, 1) \cdots C(\mu_1, k-1) \) and \( C(\mu_i, j) \) is the \( i \)-th coefficient in the Taylor expansion of \( x^n p^n_*(x) \) at \( x = 1 \).
where $C(\mu_j, i)$ denotes the binomial coefficient $\binom{p_j}{i}$, and where $A := \det [a_{i,j}], a_{i,j} = C(\mu_j, i - 1), i, j = 1, 2, \ldots, k$.

Here and below, we adopt the usual convention that $C(\mu_j, i) = 0$ if $\mu_j < i$.

Proof. Let $x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}$ denote the $k$ zeros of $p_n^*(x)$ in $(0, 1)$ (cf. Lemma 2). Then, we claim that $p_n^*(x)$ can be expressed as

$$
(2.8) \quad p_n^*(x) = \det \begin{bmatrix}
\alpha_1 & a_1^{(n)} & \cdots & a_{1,k+1}^{(n)} \\
\alpha_2 & a_2^{(n)} & \cdots & a_{2,k+1}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_k & a_k^{(n)} & \cdots & a_{k,k+1}^{(n)} \\
\alpha_k & a_k^{(n)} & \cdots & a_{k,k+1}^{(n)} \\
\end{bmatrix} \div \det \begin{bmatrix}
a_1^{(n)} & a_1^{(n)} & \cdots & a_1^{(n)} \\
a_2^{(n)} & a_2^{(n)} & \cdots & a_2^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
a_k^{(n)} & a_k^{(n)} & \cdots & a_k^{(n)} \\
a_k^{(n)} & a_k^{(n)} & \cdots & a_k^{(n)} \\
\end{bmatrix},
$$

where each entry $a_{i,j}^{(n)}$ is the $(i - 1)$-st order divided difference of the function $f_i(x) := x^\mu_j$ in the points $x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}$, i.e.,

$$
(2.9) \quad a_{i,j}^{(n)} := f_i[x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}], i = 1, \ldots, k, j = 1, \ldots, k + 1.
$$

To see this, the polynomial defined by the right side of (2.8) is evidently monic, vanishing for $x_1^{(n)}, \ldots, x_k^{(n)}$, and is thus $p_n^*(x)$ by uniqueness. Now as $n \to \infty$, we have from Lemma 2 that $x_i^{(n)} \to 1$ for each $i = 1, \ldots, k$, whence

$$
(2.10) \quad \lim_{n \to \infty} a_{i,j}^{(n)} = \frac{f_j^{(i-1)}(1)}{(i-1)!} = C(\mu_j, i - 1), i = 1, \ldots, k, j = 1, \ldots, k + 1.
$$

Furthermore, the limit of the denominator determinants in (2.8), which is $A$ from (2.10), is different from zero because it can be expressed as a nonzero constant times the Vandermonde determinant in the distinct points $\mu_1, \mu_2, \ldots, \mu_k$ (cf. [2, p. 47]). Thus, (2.7) follows from (2.10) and (2.8).

We note from (2.7) that $\lim_{n \to \infty} p_n^{(k)}(1) = k! B/A$, where $B := \det [b_{i,j}]$ with $\det [b_{i,j}] = C(\mu_j, i - 1), i, j = 1, 2, \ldots, k + 1$. Since $B$ equals $\prod_{i=0}^{k}(i1)^{-1}$ times the Vandermonde determinant in the points $\mu_1, \mu_2, \ldots, \mu_{k+1} = m$, and since $A$ equals $\prod_{i=1}^{k}(i1)^{-1}$ times the Vandermonde determinant in the points $\mu_1, \mu_2, \ldots, \mu_k$, we have that

$$
(2.11) \quad \lim_{n \to \infty} p_n^{(k)}(1) = \prod_{j=1}^{k} \frac{m - \mu_j}{(m - \mu_j)}.
$$

This fact will be useful in the
Proof of Theorem 2.1. As stated in Lemma 2, the polynomial $x^np_\ast^*(x)$ has $k + 1$ alternations in $[n^2/(n + m)^2, 1]$. Hence on replacing $x$ by $1 - t/n$, the polynomial $(1 - t/n)q_\ast(t)$, where $q_\ast(t): = p_\ast^*(1 - t/n)$, has $k + 1$ alternations in the interval

$$0 \leq t \leq n \left(1 - \frac{n^2}{(n + m)^2}\right) < 2m.$$

Furthermore, for each fixed $\rho \geq 2m$, we have (cf. (2.3))

$$n^bE_n = n^b\|x^np_\ast^*(x)\|_{[b,\rho]} = \left\|\left(1 - \frac{t}{n}\right)^n n^bq_\ast(t)\right\|_{[b,\rho]}, \quad \forall n \geq e.$$  

Next, we claim that the sequence $\{n^bq_\ast(t)\}_{n=1}^\infty$ is uniformly bounded on compact sets of the real line. Indeed, by Lemma 2, we can write

$$p_\ast^*(x) = P_\ast(x) \prod_{i=1}^k (x - x_i^{[n]}),$$

where $P_\ast(x)$ is a monic polynomial of degree $m - k$, and $n^2/(n + m)^2 < x_i^{[n]} < 1$ for each $i = 1, \cdots, k$. Thus,

$$n^bq_\ast(t) = (-1)^kP_\ast\left(1 - \frac{t}{n}\right) \prod_{i=1}^k \left(t - t_i^{[n]}\right), \quad t_i^{[n]}: = n(1 - x_i^{[n]}).$$

Next, on differentiating $k$-times the product in (2.14) via Leibniz's formula, we obtain

$$\lim_{n \to \infty} P_\ast\left(1 - \frac{t}{n}\right) = \frac{1}{k!} \lim_{n \to \infty} p_\ast^{(k)}\left(1 - \frac{t}{n}\right) = \frac{1}{k!} \lim_{a \to \infty} p_\ast^{(k)}(1),$$

uniformly on each compact subset of $\mathbb{R}$, the last equality following from (2.7). Thus, from (2.11),

$$\lim_{n \to \infty} P_\ast\left(1 - \frac{t}{n}\right) = \frac{1}{k!} \prod_{j=1}^k (m - t_j) = : \gamma,$$

so that, with (2.15), and the fact that each $t_i^{[n]} \in (0, 2m)$ it follows, as claimed, that $\{n^bq_\ast(t)\}_{n=1}^\infty$ is uniformly bounded on compact sets of the real line.

Now, let $q^\ast(t)$ be any limit function of the sequence $\{n^bq_\ast(t)\}_{n=1}^\infty$. Then by (2.15) and (2.16), $q^\ast(t)$ is a polynomial of degree $k$ of the form $q^\ast(t) = (-1)^k t^k + \cdots$. Since $-q^\ast(t)(\neq 0)$ is a limit function on $[0, \rho]$ of the sequence $\{(1 - t/n)n^bq_\ast(t)\}_{n=1}^\infty$, and since each function in this sequence has $k + 1$ alternation points in $[0, \rho]$, the same must be true of $-q^\ast(t)$. Consequently, $q^\ast(t)$ must be the unique solution of the extremal problem

$$\varepsilon(\rho): = \inf \{\|e^{-q(t)}\|_{(0,\rho)}: q(t) = (-1)^k t^k + \cdots \in \mathbb{P}_k\}.$$
But this implies that \( \{n^k q_n(t)\}_{n=1}^\infty \) has a unique limit function and, from (2.13), that
\[
\lim_{n \to \infty} n^k E_n = \| e^{-t} q(t) \|_{L_{0,1}^\infty} = \varepsilon(\rho) .
\]
Finally, since (2.18) holds for all \( \rho \geq 2m \), we have with (1.5) and (2.16) that
\[
\varepsilon(\rho) = \inf \{ \| e^{-t} q(t) \|_{L_{0,1}^\infty} : q(t) = (-1)^k \gamma t^k + \cdots \in \pi_k \}
= \gamma \varepsilon_h = \frac{\varepsilon_h}{k^\frac{b}{2}} \prod_{j=1}^k (m - \mu_j).
\]
This, coupled with (2.18), gives the desired result of (2.2).

As applications of Theorem 2.1, we now give the sharp improvements of the results in [1]. For this purpose, let \( k > 0 \) be a fixed integer, and let \( \lambda_n = (\lambda_1(n), \lambda_2(n), \cdots, \lambda_k(n)) \), where the integers \( \lambda_j(n) \) satisfy
\[
0 \leq \lambda_1(n) < \lambda_2(n) < \cdots < \lambda_k(n) < n.
\]
Then, for the extremal problem
\[
\mathcal{E}(\lambda_n) = \inf \left\{ \left\| x^* + \sum_{j=1}^k b_j x^{(j,n)} \right\|_{L_{0,1}^\infty} : (b_1, \cdots, b_k) \in \mathbb{R}^k \right\},
\]
we have

**Corollary 2.2.** If \( n - \lambda_1(n) \leq D \) and \( C \leq n - \lambda_k(n) \) for all large \( n \), where \( C, D \) are integers and \( C \geq 1 \), then
\[
\left( \frac{C + k - 1}{k} \right) \varepsilon_h \leq \lim_{n \to \infty} n^k \mathcal{E}(\lambda_n) \leq \lim_{n \to \infty} n^k \mathcal{E}(\lambda_n) \leq \left( \frac{D}{k} \right) \varepsilon_h,
\]
where \( \varepsilon_h \) is defined in (1.5). Moreover, these inequalities are best possible.

**Proof.** Since
\[
\mathcal{E}(\lambda_n) = \inf \left\{ \left\| x^* + b_{\mu_1} x^{(1,n)} + \cdots + b_{\mu_k} x^{(k,n)} \right\|_{L_{0,1}^\infty} : (b_1, \cdots, b_k) \in \mathbb{R}^k \right\},
\]
where \( \mu_j = \mu_j(n) = n + D \), then \( \mathcal{E}(\lambda_n) \) is of the form (2.1) with \( m = D \) and \( n \) replaced by \( n - D \). From the hypotheses on the \( \lambda_j(n) \), we note that \( 0 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq D - C \) for all \( n \) large. Hence, by Theorem 2.1, the set \( \mathcal{E} \) of possible limit points for the sequence \( n^k \mathcal{E}(\lambda_n) \) is given by
\[ \mathcal{L} := \left\{ \varepsilon_k \prod_{j=1}^{k} (D - \mu_j); 0 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq D - C \right\}. \]

As the largest possible limiting value occurs when \( \mu_j = j - 1, \ j = 1, \cdots, k, \) and the smallest limit occurs when \( \mu_j = D - C + j - k, \ j = 1, \cdots, k, \) the inequalities (2.20) follow.

We remark that, under the assumptions of Corollary 2.2, if \( b^\ast_j = b^\ast_j(n) \) denotes the extremal coefficients for (2.19), then Lemma 3 implies that these coefficients are bounded as \( n \to \infty. \) The precise statements of these bounds are left for the reader.

For the case when \( D = k \) and \( C = 1, \) Corollary 2.2 immediately gives

**Corollary 2.3.** For each fixed positive integer \( k, \) the quantities

\[
(2.21) \quad E_{n-k,k} = \inf \left\{ \left\| x^n + \sum_{j=1}^{k} d_j x^{n-j} \right\|_{[0,1]} : (d_1, \cdots, d_k) \in \mathbb{R}^k, \ n \geq k, \right\},
\]

satisfy

\[
(2.22) \quad \lim_{n \to \infty} n^k E_{n-k,k} = \varepsilon_k,
\]

where \( \varepsilon_k \) is defined in (1.5).

**Remark 1.** If \( d^\ast_j = d^\ast_j(n), \ j = 1, \cdots, k, \) denotes the extremal coefficients for (2.21), then since the polynomials \( x^n + \sum_{j=1}^{k} d^\ast_j x^{n-j} \) approach \( (x-1)^k \) as \( n \to \infty, \) we have

\[
(2.23) \quad \lim_{n \to \infty} d^\ast_j = (-1)^j \binom{k}{j}, \ j = 1, \cdots, k.
\]

**Remark 2.** By using a slightly different method of proof, it can also be shown that

\[
(2.24) \quad (n-k)^k E_{n-k,k} \leq \varepsilon_k, \text{ for all } n \geq k.
\]

3. Inequalities for incomplete polynomials. We now obtain estimates for the quantities \( E_{n,k} \) defined in (1.2).

**Theorem 3.1.** For any pair \((s,k)\) of integers with \( s \geq 1, \ k \geq 0, \) we have

\[
(3.1) \quad \left( \begin{array}{c} k-1/2 \\ k \end{array} \right) \left( \begin{array}{c} 2s+2k-1 \\ k \end{array} \right)^{-1} \leq E_{s,k} \leq \left( \begin{array}{c} 2s+2k \\ k \end{array} \right)^{-1}.
\]
Proof. The upper bound for $E_{s,k}$ in (3.1) follows directly from [7, eq. (30)]. To establish the lower bound in (3.1), we consider the polynomials

$$V_{s,k}(x) = \binom{2s + 2k - 1}{k}^{-1} x^s P_k^{(1/2, 1/2)}(2x - 1),$$

where $P_k^{(a,b)}(t)$ denotes, as usual, the Jacobi polynomial. From Szegö [9, p. 63], $V_{s,k}(x)$ is monic of exact degree $s + k$. It is proved in Lemmas 3.2, 3.3 of [6] that $V_{s,k}(x)$ attains positive and negative relative extrema on $(0,1]$ alternately at $k + 1$ points $\hat{\xi}_i$, with $0 < \hat{\xi}_0 < \hat{\xi}_1 < \cdots < \hat{\xi}_k = 1$, and furthermore that $\min_{0 \leq i \leq k} |V_{s,k}(\hat{\xi}_i)| = |V_{s,k}(1)|$. Thus, on regarding (1.2) as weighted best approximation from $\pi_{k-1}$ on $(0,1]$, the theorem of de la Vallée Poussin [5, p. 82] implies that $E_{s,k} \geq |V_{s,k}(1)|$. Again from Szegö [9, p. 58], $V_{s,k}(1)$ can be directly determined, so that

$$E_{s,k} \geq |V_{s,k}(1)| = \binom{2s + 2k - 1}{k}^{-1} \binom{k - 1/2}{k}.$$  

Since $\binom{k - 1/2}{k} = (2k)!/2^k(k!)^2$, we obtain, on replacing $s$ by $n - k$ in (3.1), that

$$\binom{2n}{2n-k} \frac{(2k)!}{2^{2k}(k!)^2} \leq \frac{2n}{k} E_{n-k,k} \leq 1, \quad \forall n > k.$$  

In particular, (3.2) yields

$$\binom{2k}{k} \frac{(2k)!}{2^{2k}} \leq n^k E_{n-k,k} \leq (k + 1)^k \binom{2k + 2}{k}^{-1}, \quad \forall n > k,$$

which gives explicit estimates for the positive constants $\sigma_1(k), \sigma_2(k)$ in (1.3).

Concerning bounds for the coefficients of arbitrary incomplete polynomials, we mention a simple consequence of Proposition 4 in [7]. For this purpose, we define

$$T_{s,k}(x) = Q_{s,k}(x)/E_{s,k} = x^s \sum_{i=0}^{k} \alpha_{i}^{[s,k]} x^i,$$

where, as stated in §1, $Q_{s,k}(x) \in \pi_{s,k}$ is the unique extremal polynomial for (1.2). Furthermore, let $\hat{\xi}_0^{[s,k]} < \hat{\xi}_1^{[s,k]} < \cdots < \hat{\xi}_k^{[s,k]} = 1$ denote the alternation set for $Q_{s,k}(x)$ in $[0,1]$ (cf. [7, Proposition 2]). Then, we have

Theorem 3.2. If $p(x) = x^s \sum_{i=0}^{k} \alpha_i x^i \in \pi_{s,k}$ and if $M \geq \max(|p(\hat{\xi}_i^{[s,k]})|; 0 \leq i \leq k)$, then...
\begin{equation}
|a_i| \leq M|\alpha_i|^{\nu_i}, \quad \forall \ 0 \leq i \leq k,
\end{equation}

where the $\alpha_i^{\nu_i}$ are the coefficients in (3.4).

Proof. In [7, Proposition 4], it is shown that the above hypotheses imply
\begin{equation}
\begin{aligned}
&|p(x)| \leq M|T_{\nu i}(x)|, \quad \forall x \in (\xi_i^{\nu_i}, 1), \text{ and} \\
&|p^{(\nu)}(x)| \leq M|T_{\nu i}^{(\nu)}(x)|, \quad \forall x \in (0, 1), \quad \nu = 1, 2, \ldots.
\end{aligned}
\end{equation}

Since $a_i = p^{(s+i)}(0)/(s+i)!$, and since (3.6) in particular holds when $x = 0$, the inequalities (3.5) follow. \hfill \square

4. Inequalities for polynomials on $[0, +\infty)$. As a useful analog of the first inequality of (3.6) for the interval $[0, +\infty)$, let $q_i^s(t) = t^s + \cdots$, be the unique polynomial in $\pi_k$ such that (cf. (1.5))
\begin{equation}
\varepsilon_i = \|e^{-t}q_i^s(t)\|_{[0, +\infty)},
\end{equation}

and let
\begin{equation}
0 = \eta_i^{(k)} < \eta_i^{(k-1)} < \cdots < \eta_i^{(0)}
\end{equation}

be the unique $k + 1$ alternation points for $e^{-t}q_i^s(t)$ in $[0, +\infty)$, i.e., $e^{-t}|q_i^s(t)| = \varepsilon_i$ for $t \in [0, +\infty)$ only if $t = \eta_i^{(k)}$ for some $0 \leq i \leq k$.

Then, by applying the same reasoning as in the proof of [7, Prop. 4], we have

**Theorem 4.1.** If $p \in \pi_k$ and if $M \geq \max\{|e^{-t}p(\eta_i^{(k)})|: 0 \leq i \leq k\}$, then
\begin{equation}
|p(t)| \leq M|q_i^s(t)|/\varepsilon_i,
\end{equation}

\[\forall t \in (0, \eta_i^{(k)})\hspace{1cm}.
\]

As a consequence of Theorem 4.1 and the fact (cf. (2.12) with $m = k$) that $\eta_i^{(k)} \leq 2k$, we immediately have (cf. [7, Corollary 5]) the first part of

**Theorem 4.2.** If $p \in \pi_k$ with $p \neq 0$, and if $|e^{-t}p(\zeta)| = \|e^{-t}p(t)\|_{[0, +\infty)}$, then $\zeta \leq \eta_i^{(k)} \leq 2k$; in particular,
\begin{equation}
\|e^{-t}p(t)\|_{[0, +\infty)} = \|e^{-t}p(t)\|_{[0, 2k]} = \|e^{-t}p(t)\|_{[0, k]^{(2k)}}.
\end{equation}

Moreover, the quantity $2k$ is asymptotically best possible as $k \to \infty$, in the sense that there exists a constant $\mu = 2.945 \ 820 \cdots$ for which
\begin{equation}
2k - \mu \cdot k^{1/2} + o(k^{1/2}) \leq \eta_i^{(k)} \leq 2k,
\end{equation}
as $k \to \infty$. 

Proof. To establish the result of (4.4), consider the Laguerre polynomial \( L_{\xi}^{(-1/2)}(x) \) for each \( k \geq 1 \). It is known (cf. Szegö [9, p. 176]) that \( |L_{\xi}^{(-1/2)}(0)| \) and the successive relative maxima of \( e^{-t}L_{\xi}^{(-1/2)}(2t) \) for \( t \geq 0 \) form an increasing sequence for each \( k \). Calling the abscissa of the last relative maximum \( \xi_k \), then

\[ e^{-\xi_k} |L_{\xi}^{(-1/2)}(2\xi_k)| = \|e^{-t}L_{\xi}^{(-1/2)}(2t)\|_{[0, +\infty)} , \]

which, from Theorem 4.2, implies that \( \xi_k \leq \eta_k^{(b)} \). Now, it is evident that \( 2\xi_k \) exceeds \( x_k \), the largest zero of \( L_{\xi}^{(-1/2)}(x) \), where \( x_k \) is known (cf. [9, p. 132]) to satisfy

\[ x_k = 4k - \omega k^{1/3} + o(k^{1/3}) , \quad \text{as } k \to \infty , \]

with \( \omega = 5.891639 \cdots \). But then, the inequalities \( x_k/2 < \xi_k \leq \eta_k^{(b)} \leq 2k \), coupled with (4.6), give (4.4).

For reference purposes, the numerical values of \( \eta_k^{(b)} \) of (4.1) are given in Table I below for \( 1 \leq k \leq 11 \).

We now consider estimates for the quantities \( \varepsilon_k \). Since \( n^k E_{n-k,h} \to \varepsilon_k \) as \( n \to \infty \), we deduce from the inequalities (3.2) the result of

**Theorem 4.3.** For each integer \( k \geq 0 \),

\[ \frac{(2k)!}{2^{2k}(k!)^2} \leq \frac{2^k}{k!} \varepsilon_k \leq 1 . \]

For \( k \) large, this implies, via Stirling's formula, that

\[ \frac{1}{\sqrt{\pi}} \leq \liminf_{k \to \infty} \sqrt[k]{k!} \frac{2^k}{k!} \varepsilon_k , \quad \text{while } \limsup_{k \to \infty} \frac{2^k}{k!} \varepsilon_k \leq 1 . \]

However, the upper bound in (4.8) can easily be improved. For this purpose, let \( L_{\xi}^{(a)}(t) \) denote, as usual, the Laguerre polynomial in \( \pi_k \) with respect to the weight function \( e^{-t^\alpha} \) on \([0, +\infty)\) (cf. [9, p. 100]), where \( \alpha > -1 \). Since \( L_{\xi}^{(a)}(t) = (-1)^k t^k / k! + \cdots \in \pi_k \), and since

\[ \|e^{-t^\alpha}L_{\xi}^{(a)}(t)\|_{[0, +\infty)} = \|e^{-t}L_{\xi}^{(a)}(2t)\|_{[0, +\infty)} = \|e^{-t}\left\{(-1)^k 2^k t^k + \cdots \right\}\|_{[0, +\infty)} \]

we have, by definition of \( \varepsilon_k \), that

\[ \frac{2^k}{k!} \varepsilon_k \leq \|e^{-t^\alpha}L_{\xi}^{(a)}(t)\|_{[0, +\infty)}, \quad \forall \alpha > -1 . \]

Now, in the case when \( \alpha = -1/2 \), it is known [9, p. 240] that asymptotically

\[ \|e^{-t^\alpha}L_{\xi}^{(-1/2)}(t)\|_{[0, +\infty)} \sim \frac{(18)^{1/2}}{\pi k^{3/4}} , \quad \text{as } k \to \infty , \]
where \( \tau \) is the maximum of the Airy's function \( A(t) \) for \( t \in R \). Thus, from (4.9), we have

\[
\limsup_{k \to \infty} k^{1/2} \frac{2^k}{k!} \varepsilon_k \leq (18)^{1/3} \frac{\tau}{\pi} = 0.6017 \ldots .
\]

In (4.8) and (4.11), lower and upper estimates for \( \frac{2^k \varepsilon_k}{k!} \) are given which do not have the same asymptotic behavior as \( k \to \infty \). It is an open question if there exist constants \( \alpha \) with \( 1/3 \leq \alpha \leq 1/2 \) and \( Q > 0 \) for which

\[
\lim_{k \to \infty} k^\alpha \left( \frac{2^k \varepsilon_k}{k!} \right) = Q .
\]

Finally, we include the results of numerical computations of \( \varepsilon_k \) (cf. (1.5)), using the Remez algorithm. Tabulated below in Table I are \( \frac{2^k \varepsilon_k}{k!} \) and \( \eta_k^{(b)} \) (cf. (4.1)) for \( 1 \leq k \leq 11 \), rounded to six decimal places.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \frac{2^k \varepsilon_k}{k!} )</th>
<th>( \sqrt{k} \frac{2^k \varepsilon_k}{k!} )</th>
<th>( \frac{k^{1/2} \varepsilon_k}{k!} )</th>
<th>( \frac{\eta_k^{(b)}}{\frac{(2k - \eta_k^{(b)})}{k^{1/3}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.555 929</td>
<td>.556 929</td>
<td>.556 929</td>
<td>1.278 466</td>
</tr>
<tr>
<td>2</td>
<td>.438 831</td>
<td>.613 530</td>
<td>.546 593</td>
<td>3.009 706</td>
</tr>
<tr>
<td>3</td>
<td>.369 345</td>
<td>.639 724</td>
<td>.532 688</td>
<td>4.827 187</td>
</tr>
<tr>
<td>4</td>
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<td>.655 726</td>
<td>.520 450</td>
<td>6.684 493</td>
</tr>
<tr>
<td>5</td>
<td>.298 228</td>
<td>.666 859</td>
<td>.509 963</td>
<td>8.565 402</td>
</tr>
<tr>
<td>6</td>
<td>.275 655</td>
<td>.675 215</td>
<td>.500 899</td>
<td>10.462 189</td>
</tr>
<tr>
<td>7</td>
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<td>.681 806</td>
<td>.492 959</td>
<td>12.370 440</td>
</tr>
<tr>
<td>8</td>
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<td>.687 190</td>
<td>.485 917</td>
<td>14.287 471</td>
</tr>
<tr>
<td>9</td>
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<td>.691 706</td>
<td>.479 602</td>
<td>16.211 473</td>
</tr>
<tr>
<td>10</td>
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<td>.695 570</td>
<td>.473 886</td>
<td>18.141 145</td>
</tr>
<tr>
<td>11</td>
<td>.210 735</td>
<td>.698 930</td>
<td>.468 671</td>
<td>20.075 554</td>
</tr>
</tbody>
</table>

Since \( \pi^{-1/2} = 0.564 190 \ldots \), we note that the third column of Table I is in numerical agreement with the first inequality of (4.8), while the fourth column of Table I is in numerical agreement with (4.11). Moreover, since the entries of the third and fourth columns of Table I are respectively strictly increasing and strictly decreasing, it would appear that \( 1/3 < \alpha < 1/2 \) if (4.12) were valid. Also, as a consequence of (4.4), we have

\[
2.945 \ 820 \ldots + o(1) > \frac{(2k - \eta_k^{(b)})}{k^{1/3}} \geq 0 , \text{ as } k \to \infty ,
\]

which again is in agreement with the last column on Table I.

ACKNOWLEDGMENT. We wish to thank Dr. M. Lachance (University of South Florida) for kindly performing the calculations for Table I.
Added in Proof. Using a different method the authors have recently extended Theorem 2.1 to the case of $L^q$-norms, $q \geq 1$, on $[0, 1]$.

References


Received May 31, 1979. Research by the first author was supported in part by the Air Force Office of Scientific Research, and by a Guggenheim Memorial Fellowship. Research by the second author was supported in part by the Air Force Office of Scientific Research, and by the Department of Energy.

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