

OF J. L. WALSH'S THEOREM AND ITS EXTENSION FOR INTERPOLATION IN THE ROOTS OF UNITY

E. B. SAFF¹ (Tampa) and R. S. VARGA² (Kent)

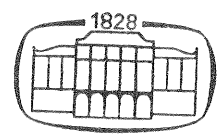
§ 1. Introduction and statements of new results

Let A_ρ denote the collection of functions analytic in $|z| < \rho$ and having a singularity on the circle $|z| = \rho$, where it is assumed that $1 < \rho < \infty$. Next, for each positive integer n , let $p_{n-1}(z; f)$ denote the Lagrange polynomial interpolant, of degree most $n-1$, of $f(z) \in A_\rho$ in the n -th roots of unity, i.e.,

$$(1.1) \quad p_{n-1}(\omega; f) = f(\omega)$$

where ω is any n -th root of unity, and let

$$(1.2) \quad P_{n-1}(z; f) := \sum_{k=0}^{n-1} a_k z^k$$



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MAISON D'EDITIONS DE L'ACADEMIE DES SCIENCES DE HONGRIE
ИЗДАТЕЛЬСТВО АКАДЕМИИ НАУК ВЕНГРИИ

A NOTE ON THE SHARPNESS OF J. L. WALSH'S THEOREM AND ITS EXTENSIONS FOR INTERPOLATION IN THE ROOTS OF UNITY

E. B. SAFF¹ (Tampa) and R. S. VARGA² (Kent)

§ 1. Introduction and statements of new results

Let A_ρ denote the collection of functions analytic in $|z| < \rho$ and having a singularity on the circle $|z| = \rho$, where it is assumed that $1 < \rho < \infty$. Next, for each positive integer n , let $p_{n-1}(z; f)$ denote the Lagrange polynomial interpolant, of degree at most $n-1$, of $f(z) \in A_\rho$ in the n -th roots of unity, i.e.,

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where ω is any n -th root of unity, and let

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be the $(n-1)$ -st partial sum of $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Letting

$$(1.3) \quad D_\tau := \{z \in \mathbb{C} : |z| < \tau\},$$

then a beautiful result of J. L. Walsh [2, p. 153] can be stated as

THEOREM A. *For each $f(z) \in A_\rho$, the interpolating polynomials of (1.1) and (1.2) satisfy*

$$(1.4) \quad \lim_{n \rightarrow \infty} \{p_{n-1}(z; f) - P_{n-1}(z; f)\} = 0, \text{ for all } z \in D_{\rho^2}.$$

Moreover, the result of (1.4) is best possible in the sense that there is some $f(z) \in A_\rho$ and some \hat{z} with $|\hat{z}| = \rho^2$ for which the sequence $\{p_{n-1}(\hat{z}; f) - P_{n-1}(\hat{z}; f)\}_{n=1}^{\infty}$ does not tend to zero as $n \rightarrow \infty$.

Note that in Theorem A, no sharpness assertions are made for *arbitrary* functions $f(z) \in A_\rho$; in particular, no statement is made on the behavior of the sequence

$$(1.5) \quad \{p_{n-1}(z; f) - P_{n-1}(z; f)\}_{n=1}^{\infty}$$

in $|z| > \rho^2$. One of the aims of this note is to in fact address this behavior in $|z| > \rho^2$. As a special case of Theorem 1 below, we prove that, for *any* $f(z) \in A_\rho$, the sequence in (1.5) can be bounded in at most *one* point in $|z| > \rho^2$. This fact is of special interest in the case when $f(z)$ in A_ρ is also continuous in the disk $|z| \leq \rho$; for such functions, it has been shown in [1, Thm. 2] that (1.4) is valid for *all* $|z| \leq \rho^2$.

¹ Research supported in part by the National Science Foundation.

² Research supported in part by the Air Force Office of Scientific Research, and by the Department of Energy.

For our own purposes below, we need a recent extension of Theorem A. For additional notation, set

$$(1.6) \quad P_{n-1,j}(z; f) := \sum_{k=0}^{n-1} a_{k+jn} z^k, \quad j = 0, 1, \dots$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 1], which gives Theorem A as the special case $l=1$, can be stated as

THEOREM B. For each $f(z) \in A_q$, and for each positive integer l , there holds

$$(1.7) \quad \lim_{n \rightarrow \infty} \left\{ p_{n-1}(z; f) - \sum_{j=0}^{l-1} P_{n-1,j}(z; f) \right\} = 0, \quad \text{for all } z \in D_{q^{l+1}},$$

the convergence being uniform and geometric on any closed subset of $D_{q^{l+1}}$. Moreover, the result of (1.7) is best possible in the sense that there is some $\tilde{f}(z) \in A_q$ and some \tilde{z} with $|\tilde{z}| = q^{l+1}$ for which the sequence

$$(1.8) \quad \left\{ p_{n-1}(z; f) - \sum_{j=0}^{l-1} P_{n-1,j}(z; f) \right\}_{n=1}^{\infty}$$

with $z = \tilde{z}$ and $f = \tilde{f}$, does not tend to zero as $n \rightarrow \infty$.

Our first new result is

THEOREM 1. For each $f(z) \in A_q$, and for each positive integer l , the sequence (1.8) can be bounded in at most l distinct points in $|z| > q^{l+1}$. This result is sharp, in the sense that, given any l distinct points $\{\eta_k\}_{k=1}^l$ in the annulus $q^{l+1} < |z| < q^{l+2}$, there is an $\hat{f}(z) \in A_q$ for which

$$(1.9) \quad \lim_{n \rightarrow \infty} \left\{ p_{n-1}(\eta_k; \hat{f}) - \sum_{j=0}^{l-1} P_{n-1,j}(\eta_k; \hat{f}) \right\} = 0, \quad k = 1, 2, \dots, l.$$

There is an extension of Theorem 1 which we can also state. Note, of course, that Theorem A involves only the Lagrange interpolation of f in the n -th roots of unity. For r a fixed positive integer, Theorem B can be extended using Hermite interpolation. For notation, let $h_{rn-1}(z; f)$ denote the Hermite polynomial interpolant, of degree at most $rn-1$, to $f, f', \dots, f^{(r-1)}$ in the n -th roots of unity, i.e.,

$$(1.10) \quad h_{rn-1}^{(j)}(\omega; f) = f^{(j)}(\omega), \quad j = 0, 1, \dots, r-1,$$

where again ω is any n -th root of unity. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, we set

$$(1.11) \quad H_{rn-1,0}(z; f) := \sum_{k=0}^{rn-1} a_k z^k,$$

and we set

$$(1.12) \quad H_{rn-1,j}(z; f) := \hat{\beta}_j(z^n) \sum_{k=0}^{n-1} a_{k+n(r+j-1)} z^k, \quad j = 1, 2, \dots,$$

where

$$(1.13) \quad \hat{\beta}_j(z) := \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z-1)^k, \quad j = 1, 2, \dots$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 3], which gives Theorem B as the special case $r=1$, can be stated as

THEOREM C. For each $f(z) \in A_\rho$, and for each pair of positive integers r and l , there holds

$$(1.14) \quad \lim_{n \rightarrow \infty} \left\{ h_{r-1}(z; f) - \sum_{j=0}^{l-1} H_{r-1,j}(z; f) \right\} = 0, \quad \text{for all } z \in D_{\rho^{1+(l/r)}},$$

the convergence being uniform and geometric for any closed subset of $D_{\rho^{1+(l/r)}}$. Moreover, the result of (1.14) is best possible in the sense that there is some $\hat{f}(z) \in A_\rho$ and some \hat{z} with $|\hat{z}| = \rho^{1+(l/r)}$ for which the sequence

$$(1.15) \quad \left\{ h_{r-1}(z; f) - \sum_{j=0}^{l-1} H_{r-1,j}(z; f) \right\}_{n=1}^{\infty},$$

with $z = \hat{z}$ and $f = \hat{f}$, does not tend to zero as $n \rightarrow \infty$.

Our second new result, which sharpens Theorem C and gives Theorem 1 as the special case $r=1$, can be stated as

THEOREM 2. For each $f(z) \in A_\rho$, and for each pair of positive integers r and l , the sequence (1.15) can be bounded in at most $r+l-1$ distinct points in $|z| > \rho^{1+(l/r)}$. This result is sharp, in the sense that, given any $r+l-1$ distinct points $\{\eta_k\}_{k=1}^{r+l-1}$ in the annulus $\rho^{1+(l/r)} < |z| < \min \left\{ \rho^{l+2}; \rho^{1+\frac{l}{r-1}} \right\}$, there is an $\tilde{f}(z) \in A_\rho$ for which

$$(1.16) \quad \lim_{n \rightarrow \infty} \left\{ h_{r-1}(\eta_k; \tilde{f}) - \sum_{j=0}^{l-1} H_{r-1,j}(\eta_k; \tilde{f}) \right\} = 0, \quad k = 1, 2, \dots, r+l-1.$$

Since the proof of Theorem 2 is completely analogous to the proof of Theorem 1, we shall give only the proof of Theorem 1.

§ 2. Proof of Theorem 1

To establish the first part of Theorem 1, consider any (fixed $f \in A_\rho$, consider any fixed positive integer l , and suppose that there are $(l+1)$ distinct points $\{y_k\}_{k=1}^{l+1}$ in $|z| > \rho^{l+1}$ for which

$$(2.1) \quad \left| p_{n-1}(y_k; f) - \sum_{j=0}^{l-1} P_{n-1,j}(y_k; f) \right| \leq M, \quad \forall n \geq 1, \quad \forall 1 \leq k \leq l+1.$$

If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then the hypothesis that f is analytic in $|z| < \rho$ with a singularity on $|z| = \rho$ gives us that

$$(2.2) \quad \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} = \frac{1}{\rho}.$$

Thus, for any $\varepsilon > 0$ with $1 < \varrho - \varepsilon$ and with

$$(2.3) \quad (\varrho - \varepsilon)^{l+2} > \varrho^{l+1},$$

there is an $n_0(\varepsilon)$ for which

$$(2.4) \quad |a_n| \leq \frac{1}{(\varrho - \varepsilon)^n}, \quad \forall n \geq n_0(\varepsilon).$$

Next, since all the points $\{y_k\}_{k=1}^{l+1}$ lie in $|z| > \varrho^{l+1}$, then

$$(2.5) \quad \varrho^{l+1} < \sigma_1 := \min_{1 \leq k \leq l+1} |y_k| \leq \max_{1 \leq k \leq l+1} |y_k| =: \sigma_2,$$

and we choose the least positive integer m for which

$$(2.6) \quad \sigma_2 < \varrho^{m+1}, \quad (\text{where } l < m).$$

Applying Theorem B (with l chosen as m), we have that the sequence

$\left\{ p_{n-1}(z; f) - \sum_{j=0}^{m-1} P_{n-1,j}(z; f) \right\}_{n=1}^{\infty}$ converges to zero for all $z \in D_{\varrho^{m+1}}$. In particular, as the points $\{y_k\}_{k=1}^{l+1}$ all lie in $D_{\varrho^{m+1}}$ from (2.5) and (2.6), then there exists a constant M_1 such that

$$(2.7) \quad \left| p_{n-1}(y_k; f) - \sum_{j=0}^{m-1} P_{n-1,j}(y_k; f) \right| \leq M_1, \quad \forall n \geq 1, \quad \forall 1 \leq k \leq l+1.$$

Using the hypothesis of (2.1), this in turn implies that

$$(2.8) \quad \left| \sum_{j=l}^{m-1} P_{n-1,j}(y_k; f) \right| \leq M_2, \quad \forall n \geq 1, \quad \forall 1 \leq k \leq l+1.$$

Recalling from (1.6) the definition of $P_{n-1,j}(z; f)$, then it follows from (2.4) that

$$|P_{n-1,j}(z; f)| \leq \sum_{k=0}^{n-1} \frac{|z|^k}{(\varrho - \varepsilon)^{k+jn}} = \frac{1}{(\varrho - \varepsilon)^{jn}} \sum_{k=0}^{n-1} \left(\frac{|z|}{\varrho - \varepsilon} \right)^k, \quad \forall n \geq n_0(\varepsilon).$$

Thus,

$$(2.9) \quad |P_{n-1,j}(z; f)| \leq \frac{n|z|^n}{(\varrho - \varepsilon)^{(j+1)n}}, \quad \forall n \geq n_0(\varepsilon), \quad \forall |z| > \varrho, \quad \forall j \geq 1.$$

This can be used as follows. From (2.9), we see that, if $l+1 \leq m-1$, then

$$(2.10) \quad \left| \sum_{j=l+1}^{m-1} P_{n-1,j}(z; f) \right| \leq \frac{(m-l-1)n|z|^n}{(\varrho - \varepsilon)^{(l+2)n}}, \quad \forall n \geq n_0(\varepsilon), \quad \forall |z| > \varrho.$$

Hence, from (2.8) and (2.10),

$$(2.11) \quad |P_{n-1,l}(y_k; f)| \leq M_2 + \frac{(m-l-1)n|y_k|^n}{(\varrho - \varepsilon)^{(l+2)n}}, \quad \forall n \geq n_0(\varepsilon), \quad \forall 1 \leq k \leq l+1.$$

Now, because of (2.11), it further follows that

$$(2.12) \quad |y_k^l P_{n,l}(y_k; f) - P_{n-1,l}(y_k; f)| \leq M_3 + \frac{M_4 n |y_k|^n}{(\varrho - \varepsilon)^{(l+2)n}},$$

for all $n \geq n_0(\varepsilon)$, all $1 \leq k \leq l+1$. Next, because of the definition of $P_{n-1,j}(z; f)$, it can be verified that

$$(2.13) \quad z^l P_{n,l}(z; f) - P_{n-1,l}(z; f) = \sum_{j=n}^{l+n} a_{ln+j} z^j - \sum_{j=0}^{l-1} a_{ln+j} z^j.$$

Obviously, the last term in (2.13) is bounded, independent of n , in the points $\{y_k\}_{k=1}^{l+1}$, whence from (2.12) and (2.13),

$$(2.14) \quad \left| \sum_{j=n}^{l+n} a_{ln+j} y_k^j \right| \leq M_5 + \frac{M_4 n |y_k|^n}{(\varrho - \varepsilon)^{(l+2)n}}.$$

On dividing through by $|y_k|^n$ in (2.14), we obtain

$$(2.15) \quad \left| \sum_{j=0}^l a_{n(l+1)+j} y_k^j \right| \leq \frac{M_5}{|y_k|^n} + \frac{M_4 n}{(\varrho - \varepsilon)^{(l+2)n}},$$

and so, from the definition of σ_1 in (2.5), there follows

$$(2.16) \quad \left| \sum_{j=0}^l a_{n(l+1)+j} y_k^j \right| \leq \frac{M_5}{\sigma_1^n} + \frac{M_4 n}{(\varrho - \varepsilon)^{(l+2)n}},$$

for all $n \geq n_0(\varepsilon)$, all $1 \leq k \leq l+1$. If, for convenience, we set

$$(2.17) \quad \tau := \max \left\{ \frac{1}{\sigma_1}; \frac{1}{(\varrho - \varepsilon)^{l+2}} \right\},$$

then it follows from (2.3) and (2.5) that

$$(2.18) \quad \tau < \frac{1}{\varrho^{l+1}}.$$

Next, we write a system of $(l+1)$ linear equations in the "unknowns" $a_{(l+1)n+j}$, i.e.,

$$(2.19) \quad \sum_{j=0}^l y_k^j a_{(l+1)n+j} =: f_{k,n}, \quad k = 1, 2, \dots, l+1$$

where, from (2.16) and (2.17),

$$(2.20) \quad |f_{k,n}| \leq M_6 n \tau^n, \quad \forall n \geq n_0(\varepsilon), \quad \forall 1 \leq k \leq l+1.$$

In matrix notation, we can write the system of equations (2.19) as

$$(2.21) \quad \begin{bmatrix} 1 & y_1 & \dots & y_1^l \\ 1 & y_2 & \dots & y_2^l \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_{l+1} & \dots & y_{l+1}^l \end{bmatrix} \cdot \begin{bmatrix} a_{(l+1)n} \\ a_{(l+1)n+1} \\ \vdots \\ a_{(l+1)n+l} \end{bmatrix} = \begin{bmatrix} f_{1,n} \\ f_{2,n} \\ \vdots \\ f_{l+1,n} \end{bmatrix}.$$

The coefficient matrix, Δ , in (2.21) is a Vandermonde matrix, and, as the points $\{y_k\}_{k=1}^{l+1}$ are *distinct* by hypothesis, then Δ is nonsingular. Using Cramer's rule, it is easy to see from (2.20) and the fact that the $\{y_k\}_{k=1}^{l+1}$ are fixed distinct points, that

$$(2.22) \quad |a_{(l+1)n+j}| \leq M_7 n \tau^n, \quad \forall n \geq n_0(\varepsilon), \quad \forall 0 \leq j \leq l.$$

However, (2.22) implies that

$$(2.23) \quad \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} \leq \tau^{1/(l+1)} < \frac{1}{\varrho},$$

the last inequality coming from (2.18). As this contradicts (2.2), then there can be at most l distinct points $\{\eta_k\}_{k=1}^l$ in $|z| > \varrho^{l+1}$ for which the sequence (1.8) is bounded, completing the first part of the proof.

To establish the second part of Theorem 1, let $w_l(z)$ be any monic polynomial of degree l with precisely l distinct zeros in the annulus $\varrho^{l+1} < |z| < \varrho^{l+2}$, i.e.,

$$(2.24) \quad w_l(z) = \prod_{k=1}^l (z - \eta_k) =: \sum_{j=0}^l \beta_j z^j,$$

where

$$(2.25) \quad \varrho^{l+1} < |\eta_k| < \varrho^{l+2} \quad \text{for } k = 1, 2, \dots, l.$$

Consider then the particular function

$$(2.26) \quad \hat{f}(z) := \frac{w_l(z)}{\varrho^{l+1} - z^{l+1}}.$$

Clearly, $\hat{f} \in A_\varrho$, and \hat{f} has $l+1$ poles on $|z| = \varrho$. We now show that with these definitions, (1.9) of Theorem 1 is satisfied. From Theorem B, we know that

$$(2.27) \quad \lim_{n \rightarrow \infty} \left\{ p_{n-1}(z; \hat{f}) - \sum_{j=0}^l P_{n-1,j}(z; \hat{f}) \right\} = 0, \quad \forall z \in D_{\varrho^{l+2}}.$$

We claim that

$$(2.28) \quad \lim_{n \rightarrow \infty} P_{n-1,l}(\eta_k; \hat{f}) = 0, \quad \forall 1 \leq k \leq l.$$

To establish (2.28), write $\hat{f}(z) := \sum_{k=0}^{\infty} \hat{a}_k z^k$. It follows from (2.24) and (2.26) that

$$(2.29) \quad \hat{a}_{m(l+1)+j} = \frac{\beta_j}{\varrho^{(m+1)(l+1)}}, \quad \forall 0 \leq j \leq l, \quad \forall m \geq 0.$$

Next, by definition,

$$(2.30) \quad P_{n-1,l}(z; \hat{f}) = \sum_{k=0}^{n-1} \hat{a}_{m+k} z^k,$$

and we consider the case when n is a multiple of $(l+1)$, i.e., $n = (l+1)s$. On regrouping terms in (2.30) for such n , $P_{n-1,l}(z; \hat{f})$ can be expressed as

$$(2.31) \quad P_{s(l+1)-1,l}(z; \hat{f}) = \sum_{k=0}^{s-1} z^{k(l+1)} \sum_{j=0}^l \hat{a}_{(l+1)[sl+k]+j} z^j.$$

But, the inner sum of (2.31) can be seen from (2.29) and (2.24) to be

$$(2.32) \quad \sum_{j=0}^l \hat{a}_{(l+1)[sl+k]+j} z^j = \frac{w_l(z)}{\varrho^{(l+1)[sl+k+1]}}.$$

Since $w_l(\eta_k)=0$ by definition, it follows from (2.31) that

$$(2.33) \quad P_{s(l+1)-1, l}(\eta_k; \hat{f}) = 0, \quad \forall 1 \leq k \leq l, \quad \forall s \equiv 1.$$

Having just considered the case when n is a multiple of $(l+1)$, we now suppose that $n=s(l+1)+t$, where $1 \leq t \leq l$. On similarly regrouping the terms in (2.30) and using the fact that $w_l(\eta_k)=0$, it can be shown that

$$(2.34) \quad P_{s(l+1)+t-1, l}(\eta_k; \hat{f}) = \sum_{j=0}^{t-1} \hat{a}_{sl(l+1)+t+j} \eta_k^j.$$

Since the $\{\eta_k\}_{k=1}^l$ are fixed, and t does not exceed l , then, as $|\hat{a}_n| \rightarrow 0$ as $n \rightarrow \infty$ from (2.29), we have from (2.33) and (2.34) that

$$(2.35) \quad \lim_{n \rightarrow \infty} P_{n-1, l}(\eta_k; \hat{f}) = 0, \quad \forall 1 \leq k \leq l,$$

as claimed in (2.28). Thus, with (2.27) and the first part of Theorem 1, the sequence

$$(2.36) \quad \left\{ p_{n-1}(z; \hat{f}) - \sum_{j=0}^{l-1} P_{n-1, j}(z; \hat{f}) \right\}_{n=1}^{\infty}$$

is convergent (to zero), only in the points $\{\eta_k\}_{k=1}^l$ and unbounded for all other points in $\{z \in \mathbb{C}: |z| > \varrho^{l+1}\}$.

Added in proof. (April 14, 1983) The second part of Theorem 1 remains valid if any l distinct points $\{\eta_k\}_{k=1}^l$ are arbitrarily chosen in $|z| > \varrho^{l+1}$, with a similar improvement holding for Theorem 2. This has been shown by the author and, more generally by T. Hermann, "Some remarks on an extension of a Theorem of Walsh", *J. Approx. Th.* (to appear).

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(Received December 27, 1981)

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