OVERRELAXATION APPLIED TO IMPLICIT ALTERNATING DIRECTION METHODS

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§1. Introduction

Let \( A = (a_{i,j}) \) be a real non-singular \( n \times n \) matrix with \( a_{i,i} \neq 0, \)
\( 1 \leq i \leq n. \) If we seek to solve the matrix equation

\[ (1) \quad Ax = k, \]

then the Young-Frankel \([1,2]\) successive overrelaxation iterative method is defined by

\[ (2) \quad x_i^{(k+1)} = x_i^{(k)} + \omega \left\{ \sum_{j \neq i} b_{i,j} x_j^{(k+1)} + \sum_{j > i} b_{i,j} x_j^{(k)} + g_i - x_i^{(k)} \right\}, \]

\[ 1 \leq i \leq n, \]

where the matrix \( B = (b_{i,j}) \), and the vector \( g \) are defined by

\[ (2') \quad b_{i,j} = \begin{cases} -a_{i,j}/a_{i,i} & \text{, } i \neq j \\ 0 & \text{, } i = j \end{cases}, \quad g_i = k_i/a_{i,i}, \quad 1 \leq i, j \leq n. \]

For suitable choice of the relaxation parameter \( \omega \), Young \([1]\) established the rapid convergence of (2) for a large class of matrix problems satisfying his definition of property (A). It was later shown that property (A) for the matrix \( A \) implies that the matrix \( B \) of (2') is, in the terminology of Romanovsky \([3]\), cyclic of index 2, and the Young-Frankel successive overrelaxation method was generalized \([4]\) to cases where the matrix \( E \) of (2') is cyclic of index \( p \geq 2. \)
Since the introduction of the Young-Frankel successive overrelaxation method, several variants of a new implicit alternating direction method \[6, 7, 8\] have appeared, and for the model problem, the numerical solution of the Dirichlet problem for the unit square on a discrete uniform mesh of side \(h\), these new iterative methods are asymptotically an order of magnitude faster than the successive overrelaxation method, as \(h \to 0\). More precisely, the rate of convergence of the successive overrelaxation method with optimum relaxation factor is, for the model problem, proportional to \(h\), while the new implicit alternating direction methods, with optimum acceleration parameters, have rates of convergence proportional to \(|\ln h|^2\) for the model problem, as \(h \to 0\). Unfortunately, these new methods have been demonstrated to be rigorously faster essentially only for the case of the Dirichlet problem for a rectangle \([5]\), since these methods depend upon the ability to split the matrix \(A\) into

\[(3) \quad A = H + V\]

where \(H\) and \(V\) are symmetric and positive definite, and

\[(4) \quad HV = VH.\]

Obviously, this implies that \(A\) is also symmetric and positive definite. For a large class of matrix equations arising from five-point numerical approximations of self-adjoint elliptic differential equations of the form

\[(5) \quad \sum_{x,y} u(x,y) - \text{div} \{D(x,y) \text{grad} u(x,y)\} = s(x,y),\]

matrix equations can be derived \([5, 8]\) so that \(A = H + V\), and all matrices are symmetric and positive definite. It is the condition of (4) which is difficult in general to fulfill.

We seek to construct an iterative method, with assumptions weaker than that of (4), whose rate of convergence for the numerical solution of the Dirichlet problem on a uniform mesh of side \(h\) in a non-rectangular region is asymptotically \(h^{1/2}\) as \(h \to 0\). We shall accomplish this by combining the attractive features of both the Peaceman-Rachford iterative method \([6]\), one particular variant of the implicit alternating direction methods, and the successive overrelaxation iterative method as applied to matrices of cyclic index greater than 2.
§2. **Cyclic Matrices**

Let $M$ be a real $r 	imes r$ matrix. If there exists an $r 	imes r$ permutation matrix $\Lambda$ such that

\[
\Lambda M \Lambda^{-1} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & M_1 \\
M_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & M_3 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & M_p \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

where the diagonal submatrices are square, we say $[3, 9]$ that $M$ is cyclic of index $p$. Such matrices are the basis for Young's property (A) and its generalizations. In fact, if $M$ is in the block form of (5), it is also consistently ordered in the sense of $[4]$. If we seek to solve the matrix equation

\[
u = M \nu + \xi,
\]

then, assuming that $M$ is block form of (5) and that the vectors $\nu$ and $\xi$ are partitioned into vector components $U_j, F_j, 1 \leq j \leq p$, according to the partitioning of the block form of (5), the successive overrelaxation iterative method applied to (6) is

\[
\begin{cases}
U_1^{(k+1)} = U_1^{(k)} + \omega \left( M_1 U_p^{(k)} + F_1 - U_1^{(k)} \right) \\
U_j^{(k+1)} = U_j^{(k)} + \omega \left( M_j U_{j-1}^{(k+1)} + F_j - U_j^{(k)} \right), \quad 2 \leq j \leq p.
\end{cases}
\]

We now consider the Peaceman-Rachford iterative method $[6]$. With (3), we write (1) as two equations

\[
\begin{cases}
(H + \rho I) \hat{x} = k + (\rho I - V) \hat{x} \\
(V + \rho I) \hat{x} = k + (\rho I - H) \hat{x}.
\end{cases}
\]

The matrices $H$ and $V$, in addition to being symmetric and positive definite, are each, after a suitable similarity transformation by a permutation matrix, tridiagonal matrices. The Peaceman-Rachford iterative method is defined by

\[
\begin{cases}
(H + \rho I) \hat{x}_1 = k + (\rho I - V) \hat{x}_1 \\
(V + \rho I) \hat{x}_{i+1} = k + (\rho I - H) \hat{x}_i,
\end{cases}
\]

where the constants $\rho$ are positive acceleration factors. The matrices $H + \rho I$ and $V + \rho I$ are symmetric and positive definite tridiagonal matrices, and thus the matrix equations of (9) can be efficiently solved by means of the Gauss elimination method $[6]$. 

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The concept of a cyclic matrix can be profitably applied to the study of the Peaceman-Rachford iterative method. We write the equations of (8) in the block form

\[
\begin{pmatrix}
H + \rho I & V - \rho I \\
H - \rho I & V + \rho I
\end{pmatrix}
\begin{pmatrix}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}
\end{pmatrix} = \begin{pmatrix}
\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}
\end{pmatrix},
\]

where \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) and \( \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \) are column vectors with 2n components. Since the square diagonal submatrices \( H + \rho I \) and \( V + \rho I \) are non-singular, (10) is equivalent to

\[
\begin{pmatrix}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}
\end{pmatrix} = \begin{pmatrix}
0 & (H + \rho I)^{-1} (\rho I - V) \\
(V + \rho I)^{-1} (\rho I - H) & 0
\end{pmatrix}
\begin{pmatrix}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}
\end{pmatrix} + \begin{pmatrix}
\begin{bmatrix} (H + \rho I)^{-1} k_1 \\ (V + \rho I)^{-1} k_2 \end{bmatrix}
\end{pmatrix}.
\]

If

\[
B_\rho \equiv \begin{pmatrix}
0 & (H + \rho I)^{-1} (\rho I - V) \\
(V + \rho I)^{-1} (\rho I - H) & 0
\end{pmatrix}, \quad u \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

then (11) is exactly of the form (6), where \( B_\rho \) is cyclic of index 2. Applying the Gauss-Seidel iterative method to (6), the special case \( \omega = 1 \) of (7), to the matrix equation (6) with \( B_\rho \equiv M \), we have

\[
\begin{cases}
x_1^{(\ell + 1)} = (H + \rho I)^{-1} (\rho I - V) x_2^{(\ell)} + (H + \rho I)^{-1} k_2 \\
x_2^{(\ell + 1)} = (V + \rho I)^{-1} (\rho I - H) x_1^{(\ell)} + (V + \rho I)^{-1} k_2,
\end{cases}
\]

or equivalently

\[
\begin{cases}
(H + \rho I) x_1^{(\ell + 1)} = (\rho I - V) x_2^{(\ell)} + k_1 \\
(V + \rho I) x_2^{(\ell + 1)} = (\rho I - H) x_1^{(\ell)} + k_2.
\end{cases}
\]

This, except for notation, is equivalent to (9) for the case in which all the
parameters $\rho_1$ are constant. Thus, the Peaceman-Rachford iterative method with parameters $\rho_1$ all constant is just the Gauss-Seidel iterative method applied to a matrix which is cyclic of index 2. Generalizing, if the Peaceman-Rachford iterative scheme uses $q$ parameters $\rho_1$ cyclically, then we consider

$$
u = B_{\rho}(\nu) = \nu + z,$$

where

$$B_{\rho}(q) = \begin{pmatrix}
0 & 0 & \cdots & 0 & (H + \rho_1 I)^{-1}(\rho_1 I - V) \\
(V + \rho_1 I)^{-1}(\rho_1 I - H) & 0 & 0 & 0 \\
0 & (H + \rho_2 I)^{-1}(\rho_2 I - V) & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & (V + \rho_q I)^{-1}(\rho_q I - H) \end{pmatrix}.$$ (16)

Thus, $B_{\rho}(q)$ is cyclic of index $2q$, and $\nu$ is a vector with $2n \cdot q$ components. It is easily shown that the Gauss-Seidel iterative method applied to (15) is equivalent to the Peaceman-Rachford iterative method with $q$ parameters $\rho_1$ used cyclically. The special case $\rho_1 = \rho_2 = \cdots = \rho_q = \rho$ is admitted in the preceding formulas.

It is interesting to point out that the Gauss-Seidel $L(\rho)$ matrix derived from (13) is

$$L(\rho) = \begin{pmatrix}
0 & (H + \rho_1 I)^{-1}(\rho_1 I - V) \\
0 & (V + \rho_1 I)^{-1}(\rho_1 I - H)(H + \rho_1 I)^{-1}(\rho_1 I - V)
\end{pmatrix},$$ (18)

so that we may write (13) in the form

$$\begin{pmatrix}
\bar{x}_1(\lambda + 1) \\
\bar{x}_2(\lambda + 1)
\end{pmatrix} = L(\rho) \begin{pmatrix}
\bar{x}_1(\lambda) \\
\bar{x}_2(\lambda)
\end{pmatrix} + \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}.$$ (13')

The eigenvalues of $L(\rho)$ are thus zeros and the eigenvalues of the matrix $T_\rho \equiv (V + \rho_1 I)^{-1}(\rho_1 I - H)(H + \rho_1 I)^{-1}(\rho_1 I - V)$, which we shall call the Peaceman-Rachford matrix.

We seek to obtain rates of convergence of order $h^{1/2}$, for the Dirichlet problem on a uniform mesh, as follows. The matrix $B_\rho(2)$, which depends on but one parameter $\rho$, is cyclic of index 4 and consistently ordered. By choosing the parameter $\rho$ so as to minimize the spectral radius of $B_\rho(2)$, we then apply
the successive overrelaxation theory of [4] to the matrix equation (15), with 
\( B_\rho(2) \) replacing \( B_\rho(q) \). This introduces a second parameter \( \omega \). The conditions 
under which successive overrelaxation can be rigorously applied to (15) are 
weaker than the condition of (4), and includes (4) as a special case.

§3. One Parameter Acceleration Methods

Both the successive overrelaxation iterative method, with relaxation 
factor \( \omega \), and the Peaceman-Rachford iterative method, with a single parameter \( \rho \), 
can be considered as one parameter iterative methods. It is of interest to 
compare their rates of convergence. The spectral radius of the Peaceman-Rachford 
matrix

\[
T_\rho = (V + \rho I)^{-1} (\rho I - H) (H + \rho I)^{-1} (\rho I - V)
\]

(18)

can be estimated as follows. Since \( H \) and \( V \) are symmetric and positive definite 
matrices, let \( 0 < \alpha_1 \leq x \leq \beta_1 \), \( 0 < \alpha_2 \leq x \leq \beta_2 \) be the smallest intervals containing 
all the eigenvalues of \( H \) and \( V \), respectively. We now consider the matrix \( \hat{T}_\rho \)
declared by

\[
\hat{T}_\rho = (V + \rho I) T_\rho (V + \rho I)^{-1} = (\rho I - H) (H + \rho I)^{-1} (\rho I - V) (V + \rho I)^{-1}
\]

(19)

If \( \| x \| \equiv \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \), for \( \rho > 0 \), let

\[
\| \rho I - V \| (V + \rho I)^{-1} \| \equiv \Lambda_1 (\rho).
\]

If \( \| x \| = 1 \)

Since \( V \) is symmetric and positive definite, then \( x = \sum_{k=1}^{n} c_k \xi_k \), where the 
\( \xi_k \) form an orthonormal basis for our \( n \)-dimensional vector space over the complex 
numbers, and \( V \xi_k = \lambda_k \xi_k \). Thus

\[
(\rho I - V) (V + \rho I)^{-1} x = \sum_{k=1}^{n} c_k \left( \frac{\rho - \lambda_k}{\rho + \lambda_k} \right) \xi_k.
\]

Since \( \max_k \left( \frac{\rho - \lambda_k}{\rho + \lambda_k} \right) \leq \max_{\alpha_2} \left( \frac{\rho - \alpha_2}{\rho + \alpha_2} \right) \leq \max_{\beta_2} \left( \frac{\rho - \beta_2}{\rho + \beta_2} \right) \),

\[
\sum_{k=1}^{n} |c_k|^2 = 1 \text{ from } \| x \| = 1, \text{ it follows that}
\]
(21) \[ \Lambda_1 (\rho) \leq \max \left\{ \frac{|\rho - \alpha_2|}{|\rho + \alpha_2|}, \frac{|\rho - \beta_2|}{|\rho - \beta_2|} \right\} \leq \tilde{\Lambda}_1 (\rho) < 1. \]

Similarly, if

(20') \[ \sup_{\|x\| = 1} \| \left( \rho I - H \right) \left( H + \rho I \right)^{-1} x \| \equiv \Lambda_2 (\rho), \]

then

(21') \[ \Lambda_2 (\rho) \leq \max \left\{ \frac{|\rho - \alpha_1|}{|\rho + \alpha_1|}, \frac{|\rho - \beta_1|}{|\rho - \beta_1|} \right\} \equiv \tilde{\Lambda}_2 (\rho) < 1. \]

It follows now that \( \| \tilde{T}_\rho x \| \leq \Lambda_1 (\rho) \cdot \Lambda_2 (\rho) \), and since \( \tilde{T}_\rho \) and \( T_\rho \) have the same eigenvalues,

(22) \[ \bar{\mu} (T_\rho) \leq \Lambda_1 (\rho) \cdot \Lambda_2 (\rho) \leq \tilde{\Lambda}_1 (\rho) \cdot \tilde{\Lambda}_2 (\rho) < 1. \]

Thus, \( \bar{\mu} (T) \), the spectral radius\(^\text{a}\) of the matrix \( T \), is less than unity, showing that \( T \) is a convergent matrix \([5]\). If

(23) \[ \min_{\rho > 0} \tilde{\Lambda}_1 (\rho) \cdot \tilde{\Lambda}_2 (\rho) = \tilde{\Lambda}_1 (\rho_0) \cdot \tilde{\Lambda}_2 (\rho_0), \]

it can be shown, with \( \rho_1 = \sqrt{\alpha_1 \beta_1} \) and \( \rho_2 = \sqrt{\alpha_2 \beta_2} \), that

(24) \[ \max \left\{ \rho_1, \rho_2 \right\} \geq \rho_0 \geq \min \left\{ \rho_1, \rho_2 \right\}. \]

For the numerical solution of the Dirichlet problem on the unit square with uniform mesh spacing \( h = 1/N \), the five point approximation

(25) \[ 4u_{i,j} - (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) = 0 \quad , \quad 1 \leq i, j \leq N-1, \]

to Laplace's equation leads to a system of \((N-1)^2\) linear equations. It can be verified that the spectral radius of the corresponding Jacobi matrix \( B \) of (2') is

\(^{a}\)If \( C \) is an arbitrary \( m \times n \) complex matrix, then \( \bar{\mu} (C) \equiv \max_k |\lambda_k| \), where the \( \lambda_k \) are eigenvalues of \( C \).
(26) \( \mu(B) = \cos(\varpi h) \).

Assuming that the matrix \( A \) of (1) is consistently ordered, the formula for the optimum relaxation factor \( \omega_b \) is

\[
(27) \quad \omega_b = \frac{2}{1 + \sqrt{1 - \frac{2}{\mu(B)}}} = \frac{2}{1 + \sin(\varpi h)}.
\]

If \( R[\mu] = -\ln \mu(\omega) \) is the rate of convergence for the successive overrelaxation method, it is known \([1]\) that \( \mu(\omega_b) = (\omega_b - 1) \) and asymptotically

\[
(28) \quad R[\omega_b] = 2\varpi h + O(h^2), \quad h \to 0.
\]

For this model problem, \( a_1 = a_2 = 2 \left[ 1 - \cos(\varpi h) \right], \beta_1 = \beta_2 = 2 \left[ 1 + \cos(\varpi h) \right] \), and \( \rho_1 = \rho_2 = 2 \sin(\varpi h) \). From (24), \( \rho_0 = 2 \sin(\varpi h) \), and from (22), we derive

\[
(29) \quad \mu(T_0) \leq \left[ \frac{1 + \cos(\varpi h) - \sin(\varpi h)}{1 + \cos(\varpi h) + \sin(\varpi h)} \right]^2 \leq \omega_b - 1.
\]

It can actually be shown that \( \min_\rho \mu(T_0) = \mu(T_0) = \omega_b - 1 \), and thus

\[
(30) \quad \max_\rho R[T_0] = R[T_0] = R[\omega_b] = \max_\omega R[\omega].
\]

We conclude that, for the model problem, the Peaceman-Rachford iterative method, with a single optimum parameter \( \rho \), is exactly as fast as the successive overrelaxation method with optimum relaxation parameter \( \omega \). Thus, from (28), both of these one parameter methods give rates of convergence proportional to \( h \), the mesh spacing, as \( h \to 0 \).

We now consider the numerical solution of the Dirichlet problem for a closed set of points \( \Omega \) consisting of the (connected) union of a finite number of unit squares with sides parallel to the coordinate axes in the plane. We denote by \( \Omega \) the smallest (in area) square consisting of unit squares such that \( \Omega \subset \Omega \). If we impose a uniform mesh of side \( h = 1/N \) over \( \Omega \) and \( \Omega \), and approximate Laplace's equation as in (25), then the eigenvalue bounds \( a_i, \beta_i \) and \( \alpha_i, \beta_i \) for \( \Omega \) and \( \Omega \) respectively satisfy

\[
(31) \quad \left\{ \begin{array}{l}
0 < \alpha_i \leq a_i \leq \beta_i \leq \beta_i \\
\sqrt{\alpha_i} \leq \sqrt{a_i} \leq \beta_i \leq \beta_i 
\end{array} \right\}, \quad i = 1, 2.
\]
where $\alpha_1 = \alpha_2$, $\rho_1 = \rho_2$. Let $\bar{\mu}(T_\rho(\Omega))$ and $\bar{\mu}(T_\rho(\Omega))$ denote respectively the spectral radii of the Peaceman-Rachford matrix with a single parameter for the numerical solution of the Dirichlet problem over $\Omega$ and $\Omega$. Applying the previous results of the model problem to $\Omega$, we conclude that

\begin{equation}
\bar{\mu}(T_{\rho}(\Omega)) = \left(\frac{\rho_1 - \sqrt{\alpha_1 \rho_1}}{\rho_1 + \sqrt{\alpha_1 \rho_1}}\right)^2 = \min_{\rho} \bar{\mu}(T_\rho(\Omega)),
\end{equation}

where $\rho_0 = \sqrt{\alpha_1 \rho_1} = \sqrt{\alpha_2 \rho_2}$. Because of the inequalities of (30), we have, for $\bar{\mu}(\Omega)$, that

\begin{equation}
\bar{\lambda}_1(\rho_0) = \left(\frac{\rho_1 - \rho_0}{\rho_1 + \rho_0}\right) \leq \left(\frac{\rho_1 - \rho_0}{\rho_1 + \rho_0}\right) \leq \left(\frac{\rho_2 - \rho_0}{\rho_2 + \rho_0}\right).
\end{equation}

Since $\rho_1 = \rho_2$, then with (22) and (31), we have

\begin{equation}
\bar{\mu}(T_{\rho}(\Omega)) \leq \bar{\mu}(T_{\rho}(\Omega)),
\end{equation}

and thus

\begin{equation}
\max_{\rho} R(T_{\rho}(\Omega)) \geq \max_{\rho} R(T_{\rho}(\Omega)) = R(T_{\rho_0}(\Omega)).
\end{equation}

We conclude that the optimum rate of convergence, for the numerical solution of the Dirichlet problem, of the Peaceman-Rachford iterative method with a single parameter $\rho$ is at least as fast for $\Omega$ as it is for $\Omega$. Young and Ehrlich observed this numerically in [11]. What is more important, however, is that for either $\Omega$ or $\Omega$, the rate of convergence of the optimized Peaceman-Rachford method with a single parameter is at least of order $h$, as $h \to 0$.

§4. Results of the $p$-Cyclic Theory and Applications

We briefly summarize the pertinent results of [4]. Let $B$ be a consistently ordered real $n \times n$ matrix which is cyclic of index $p \geq 2$. If $1 > \bar{\mu}(B) > 0$, let $\omega_b$ be the unique positive value (less than $p/(p - 1)$) satisfying

\begin{equation}
\omega_b = \left(\frac{\omega}{\bar{\mu}(B)}\right)^p (p - 1) \quad (\omega - 1).
\end{equation}
From the mapping \( \mu(Z) = \frac{1}{\omega_b} \left\{ Z + \frac{\omega_b - 1}{z^{p-1}} \right\} \), let \( S_p(\overline{\mu}(B)) \) be the image of the circle \( |Z| = \left| (\omega_b - 1)(p - 1) \right|^{1/p} \). For \( p = 2 \), \( S_p \) is the interval \( -\overline{\mu}(B) \leq x \leq -\overline{\mu}(B) \). For \( p > 2 \), \( S_p \) is a Jordan curve whose interior contains the origin \( \mu = 0 \). Let \( S_p(\overline{\mu}(B)) \) denote the set of points \( \mu \) consisting of \( S_p \) and its interior. If \( \Lambda_\omega \) is the overrelaxation matrix derived from \( B \) where \( \omega \) is the relaxation factor, then [4] if the eigenvalues \( \mu \) of \( A \) lie in \( S_p(\overline{\mu}(B)) \),

\[
\overline{\mu}(\Lambda_\omega) = (\omega_b - 1)(p - 1)
\]

(34)

\[
\overline{\mu}(\Lambda_\omega) > \overline{\mu}(\Lambda_{\omega_b}) \quad \text{for} \; \omega \neq \omega_b.
\]

Now let \( T_p(\overline{\mu}(B)) \) be the set of points \( \mu \) where \( \mu \in S_p(\overline{\mu}(B)) \). For \( p = 2 \), \( T_2 \) is the interval \( 0 \leq x \leq -\overline{\mu}^2 B \). But for \( p > 2 \), \( T_p \) not only contains the interval \( 0 \leq x \leq -\overline{\mu}^p(B) \), but non-real points as well. Since the relationship [4] between the eigenvalues \( \mu \) of \( A \) and the eigenvalues \( \lambda \) of \( \Lambda_1 \), the Gauss-Seidel matrix, is \( \lambda = \mu^p \), then the results of (34) hold if the eigenvalues of \( \Lambda_1 \) lie in \( T_p(\overline{\mu}(B)) \). From [4], we have the following asymptotic relationship

\[
R(\Lambda_{\omega_b}) \approx \left( \frac{2p}{p - 1} \right)^{1/2} \left[ R(\Lambda_1) \right]^{1/2}, \quad \mu(3) \rightarrow 1^{-}.
\]

With these facts, we come to the application of the above theory to the iterative method which combines successive overrelaxation and the Peaceman-Rachford iterative method. In §2, we showed that the Peaceman-Rachford iterative method with two parameters \( \rho_1 \) and \( \rho_2 \) was equivalent to applying the Gauss-Seidel iterative method to a particular 4-cyclic matrix. With \( \rho_2 = \rho_1 = \rho \), let \( \Lambda_1(\rho) \) denote the Gauss-Seidel matrix whose spectral radius is \( \Lambda_1(\rho) \). With the above discussion, we have the

**Theorem.** If the eigenvalues of \( \Lambda_1(\rho) \) lie in \( T_4(\overline{\mu}(\rho)) \), then the optimum relaxation factor \( \omega_b \) satisfies (33) with \( p = 4 \), and

\[
\overline{\mu}(\Lambda_{\omega_b}(\rho)) = 3(\omega_b - 1).
\]

**Corollary.** If \( H \) and \( V \) satisfy (4), then the eigenvalues of \( \Lambda_1(\rho) \) lie in \( T_4(\overline{\mu}(\rho)) \).
for all $\rho > 0$, and the conclusions of the above theorem are valid.

The proof of the corollary follows from the fact that the eigenvalues of $L_1(\rho)$ are just zeros and the eigenvalues of $T_0^2$. But if $H$ and $V$ commute, the eigenvalues of $L_1(\rho)$ are necessarily non-negative real numbers for every $\rho > 0$, and we can apply the results of the theorem above.

Finally, this new iterative method, which depends on two acceleration parameters $\rho, \omega$, has a rate of convergence of order $h^{1/2}$, as $h \to 0$, for certain problems. To see this, we obtained a rate of convergence of order $h$ for the Dirichlet problem of $\S 3$ without the assumption of (4), by selecting the parameter $\rho$ properly. Assuming the weaker conditions of the Theorem, we can apply the asymptotic results of (35), and conclude that our two parameter iterative method with optimum parameters is of order $h^{1/2}$ as $h \to 0$, for the solution of the Dirichlet problem in the plane.
Bibliography


