A Lower Bound for the de Bruijn-Newman Constant $\Lambda^*$

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Dedicated to the memory of our good friend Peter Henrici
(Sept. 13, 1923–March 13, 1987)

Summary. A lower bound is constructively found for the de Bruijn-Newman constant $\Lambda$, which is related to the Riemann Hypothesis. This lower bound is determined by explicitly exhibiting an associated Jensen polynomial with nonreal zeros.

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Introduction

The purpose of this paper is to obtain a constructive finite lower bound for the de Bruijn-Newman constant $\Lambda$, which is related to the Riemann Hypothesis. To our knowledge, no such lower bound for $\Lambda$ appears in the literature.

For background, if $\zeta(z)$ denotes the Riemann zeta function, it is known that all zeros of $\zeta(z)$, except for simple zeros at $-2, -4, -6, \ldots$, lie in the “critical” strip defined by $0 \leq \text{Re} \, z \leq 1$ (cf. Titchmarsh [15, p. 30]), and the Riemann Hypothesis is the statement that all the zeros of $\zeta(z)$ in this critical strip lie precisely on the line $\text{Re} \, z = 1/2$. With Riemann’s definition of his $\xi$-function, i.e.,

$$\xi(iz) = \frac{1}{2} \left( z^2 - \frac{1}{4} \right) \pi^{-\frac{3}{2}} 4^{-\frac{1}{2}} \Gamma \left( \frac{z + 1}{2} \right) \zeta \left( \frac{z + 1}{2} \right),$$

(1)

it is known (cf. Henrici [5, p. 305]) that the Riemann Hypothesis is equivalent to the statement that all zeros of $\xi(z)$ are real. It is further known that $\xi$ is an entire function of order one (cf. [15, p. 29]), and admits the integral representation (cf. Pólya [10, p. 11] or [15, p. 255])

$$\xi \left( \frac{x}{2} \right) = 8 \int_0^\infty \Phi(t) \cos(x \, t) \, dt,$$

(2)

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where
\[ \Phi(t) := \sum_{n=1}^{\infty} \frac{(2n^4 \pi^2 e^{9t} - 3n^2 \pi e^{5t}) \exp(-n^2 \pi e^{4t})}{n} \quad (0 \leq t < \infty). \]  
(3)

It is also known (cf. Pólya [10] or Csordas, Norfolk, and Varga [4, Theorem A]) that \( \Phi \) satisfies the following properties:

i) \( \Phi(z) \) is analytic in the strip \( -\pi/8 < \text{Im} \, z < \pi/8; \)

ii) \( \Phi(t) = \Phi(-t), \) and \( \Phi(t) > 0 \) \( (t \in \mathbb{R}); \)

iii) for any \( \varepsilon > 0, \lim_{t \to -\infty} \Phi^{(n)}(t) \exp[(\pi - \varepsilon) e^{4t}] = 0, \) \( \) for each \( n = 0, 1, 2, \ldots. \)

(4)

From (2), the Taylor series for \( \zeta \) is given by
\[ \frac{1}{8} \zeta \left( \frac{x}{2} \right) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m x^{2m}}{(2m)!}, \]
(5)

where
\[ \hat{b}_m := \int_0^\infty t^{2m} \Phi(t) \, dt \quad (m = 0, 1, 2, \ldots). \]
(6)

(Note from (4ii) that \( \hat{b}_0 > 0, \) so that \( \zeta(0) \neq 0. \) On setting \( z = -x^2 \) in (5), the function \( F_0(z) \), defined by
\[ F_0(z) := \sum_{m=0}^{\infty} \frac{\hat{b}_m z^m}{(2m)!}, \]
(7)

is then an entire function of order 1/2. Furthermore, from these definitions, \( x_0 \) is a real zero of \( \zeta \left( \frac{x}{2} \right) \) iff \( z_0 := -x_0^2 \) is a negative real zero of \( F_0(z) \). From this, it follows that the Riemann Hypothesis is equivalent to the statement that all zeros of \( F_0(z) \) are real and negative.

Continuing, let \( H_0(x) \) be the Fourier transform of \( \Phi(t)/2, \) i.e., (cf. (4ii))
\[ H_0(x) := \frac{1}{2} \int_{-\infty}^{\infty} \Phi(t) e^{ixt} \, dt = \int_0^{\infty} \Phi(t) \cos(xt) \, dt, \]
(8)

so that \( H_0(x) = \zeta(x/2)/8. \) Based on the properties of the function \( \Phi(t) \) in (4), it can be shown, for any real \( \lambda, \) that the function \( H_\lambda(x), \) defined by
\[ H_\lambda(x) := \frac{1}{2} \int_{-\infty}^{\infty} e^{\lambda t^2} \Phi(t) e^{ixt} \, dt = \int_0^{\infty} e^{\lambda t^2} \Phi(t) \cos(xt) \, dt \quad (\lambda \in \mathbb{R}), \]
(9)

is an entire function of order one, with \( H_\lambda(x) \) real for real \( x. \) More precisely, the sharper result is that \( H_\lambda(x) \) is an entire function of order 1 and of type \( \infty \) for any \( \lambda \in \mathbb{R}. \) Because this sharper result is not needed in what immediately follows, we have given the proof of this assertion in Appendix A.

Continuing, as a special case of a result of Pólya [11], if \( H_0(x) \) has only real zeros, then the entire function \( H_\lambda(x) \) also has only real zeros, provided
that \( \lambda \geq 0 \). Now, two results of de Bruijn [2] establish that

i) \( H_\lambda(x) \) has only real zeros for any \( \lambda \geq 1/2 \);

ii) if \( H_\lambda(x) \) has only real zeros for some real \( \lambda \), then \( H_{\lambda'}(x) \) has only real zeros for any \( \lambda' \geq \lambda \).

(10)

In particular, it follows from (10) that if the Riemann Hypothesis is true, then \( H_\lambda(x) \) must possess only real zeros for any \( \lambda \geq 0 \).

It is interesting that Newman [7] has shown more recently that there exists a real number \( A \), which in our notation satisfies \(-\infty < A \leq 1/2\), such that (cf. (9))

\[
H_\lambda(x) \text{ has only real zeros when } \lambda \geq A, \text{ and}
\]

\[
H_\lambda(x) \text{ has some nonreal zeros when } \lambda < A.
\]

(11)

Because of de Bruijn's earlier related work, it seems fitting to call the above constant \( A \) the \textit{de Bruijn-Newman constant}.

We remark that Newman's method of proof for the existence of \( A \) in [7] is nonconstructive, and no finite lower bound for \( A \) appears in [7], or in any publications that we are aware of. Of course, from our discussion above, the truth of the Riemann Hypothesis would imply that \( A \leq 0 \), while Newman [7] offers the complementary conjecture that this de Bruijn-Newman constant \( A \) satisfies \( A \geq 0 \). Because of this connection of \( A \) to the Riemann Hypothesis, obtaining constructive lower bounds for \( A \) seems desirable.

Returning to \( H_\lambda(x) \) of (9), it follows, in analogy with (5), that

\[
H_\lambda(x) = \sum_{m=0}^{\infty} (-1)^m \bar{b}_m(\lambda) \frac{x^{2m}}{(2m)!},
\]

(12)

where

\[
\bar{b}_m(\lambda) := \int_0^\infty t^{2m} e^{t^2} \phi(t) \, dt \quad (m=0, 1, 2, \ldots; \lambda \in \mathbb{R}).
\]

(13)

On setting \( z = -x^2 \) in (12), we obtain the entire function

\[
F_\lambda(z) := \sum_{m=0}^{\infty} \frac{\bar{b}_m(\lambda) z^m}{(2m)!} \quad (\lambda \in \mathbb{R}),
\]

(14)

which is of order 1/2, and (14) extends the definition of (7) to all real values of \( \lambda \). Noting that \( \bar{b}_m(\lambda) > 0 \) from (13) and (4ii) for all \( m \geq 0 \), then as before, \( x_0(\lambda) \) is a real zero of \( H_\lambda(x) \) iff \( z_0(\lambda) = -x_0^2(\lambda) \) is a negative real zero of \( F_\lambda(z) \).

Thus, all the zeros of \( H_\lambda(x) \) are real iff all the zeros of \( F_\lambda(z) \) are real and negative, and from (11), the de Bruijn-Newman constant \( A \) can be equivalently determined by the conditions

\[
F_\lambda(z) \text{ has only real negative zeros when } \lambda \geq A, \text{ and}
\]

\[
F_\lambda(z) \text{ has some nonreal zeros when } \lambda < A.
\]

(15)
We next remark that since \( F_\lambda(z) \) is a transcendental (i.e., non-polynomial) entire function of order \( 1/2 \) for any real \( \lambda \), it can be expressed (using the Hadamard factorization theorem) as

\[
F_\lambda(z) = C(\lambda) \prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n(\lambda)} \right),
\]

(16)

where from (14), \( C(\lambda) = \delta_0(\lambda) > 0 \), and where \( |z_n(\lambda)| > 0 \) for all \( n \geq 1 \), with

\[
\sum_{n=1}^{\infty} |z_n(\lambda)|^{-1} < \infty.
\]

In particular, if the real number \( \lambda \) is such that \( z_n(\lambda) < 0 \) for all \( n \geq 1 \), then \( F_\lambda(z) \) is seen to be an element of the Laguerre-Pólya class (written \( F_\lambda \in \mathcal{L} - \mathcal{P} \)), this class being defined as the collection of all entire functions \( f(x) \) of the form

\[
f(x) = C e^{-x^2 + \beta x} x^{\alpha} \prod_{j=1}^{\infty} \left( 1 - \frac{x}{x_j} \right)^{e^{x_j x}} \quad (\omega \leq \infty),
\]

(17)

where \( x \geq 0 \), \( \beta \) and \( C \) are real numbers, \( n \) is a nonnegative integer, and the \( x_j \)'s are real and nonzero with \( \sum_{j=1}^{\infty} 1/x_j^2 < \infty \). Thus, we can express (15) alternately as

\[
F_\lambda \in \mathcal{L} - \mathcal{P} \text{ when } \lambda \geq \Lambda, \text{ and}
\]

\[
F_\lambda \notin \mathcal{L} - \mathcal{P} \text{ when } \lambda < \Lambda.
\]

(18)

Next, it is convenient to set

\[
u_m(\lambda) = (m!) \delta_m(\lambda)/(2m)! \quad (m = 0, 1, 2, \ldots; \lambda \in \mathbb{R}),
\]

(19)

so that, from (14), we can write

\[
F_\lambda(z) = \sum_{j=0}^{\infty} \frac{\underline{u}_m(\lambda) z^m}{m!} \quad (\lambda \in \mathbb{R}).
\]

(20)

The \( n \)th Jensen polynomial associated with \( F_\lambda(z) \) is then defined as

\[
G_n(t; \lambda) := \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) u_k(\lambda) t^k \quad (n = 0, 1, 2, \ldots; \lambda \in \mathbb{R}).
\]

(21)

Now, it is known (cf. Pólya-Schur [12]) that \( F_\lambda(z) \in \mathcal{L} - \mathcal{P} \) iff \( G_n(t; \lambda) \in \mathcal{L} - \mathcal{P} \) \((n = 0, 1, 2, \ldots)\). As a consequence, if \( F_\lambda(z) \in \mathcal{L} - \mathcal{P} \) then the zeros of each \( G_n(t; \lambda) \) must be all real and negative for any \( n \geq 1 \). From this and (18), we immediately have

**Proposition 1.** Suppose that there is a positive integer \( m \) and a real number \( \lambda \) such that \( G_m(t; \lambda) \) possesses a nonreal zero. Then,

\[
\lambda < \Lambda,
\]

(22)

where \( \Lambda \) is the de Bruijn-Newman constant.

This will be used in the next section.
2. Determination of a Lower Bound for \( A \)

We can use Proposition 1 in a numerically rigorous fashion to obtain a constructive finite lower bound for \( A \). The idea is the following. For any real \( \lambda \), the associated moments \( \{ \beta_n(\lambda) \}_{n=0}^{\infty} \), from (13), can be computed with a known precision, as was similarly done in [4]. Calling the approximate moments \( \{ \beta'_n(\lambda) \}_{n=0}^{\infty} \), the Jensen polynomials associated with these approximate moments are then defined (cf. (19) and (21)) by

\[
g_n(t; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \frac{(k!) \beta'_k(\lambda)}{(2k)!} t^k \quad (n = 1, 2, \ldots, N),
\]

which differ, because of errors in the \( \beta'_n(\lambda) \)'s, from the exact Jensen polynomials \( G_n(t; \lambda) \) of (21). One then finds the zeros of the \( \{ g_n(t; \lambda) \}_{n=1}^{\infty} \) with a good high-precision root finder, such as the Jenkins algorithm [5].

To estimate the discrepancies between the zeros of \( g_n(t; \lambda) \) and \( G_n(t; \lambda) \), we make use of the following well-known result of Ostrowski [8] and [9, Appendix B]:

**Proposition 2.** Let \( f(z) = \sum_{j=0}^{n} a_j z^j \) and \( h(z) = \sum_{j=0}^{n} b_j z^j \) be complex polynomials with \( a_0, a_\neq 0 \), and let the zeros of \( f(z) \) be \( \{ z_j \}_{j=1}^{n} \) (which are all nonzero, but multiple zeros can occur). Assume that there is a real number \( \tau \) with \( 4n \tau^{1/n} \leq 1 \), such that

\[
|b_j - a_j| \leq \tau |a_j| \quad (0 \leq j \leq n).
\]

Then, the \( n \) zeros \( \{ z_j \}_{j=1}^{n} \) of \( h(z) \) can be ordered in such a way that

\[
\left| \frac{z_j}{x_j} - 1 \right| < 8n \tau^{1/n} \quad (1 \leq j \leq n).
\]

Our main result is the

**Theorem.** Let \( A \) be the \( a \)e Bruijn-Newman constant of (11). Then,

\[-50 < A.\]

**Remark.** In our opinion, the importance of (26) is that a constructive method has been found to produce a finite lower bound for \( A \), a bound which can be checked independently by others. No special significance is attached to the number \(-50\). In fact, it can be expected that our lower bound can be substantially increased, at the expense of using even higher precision calculations of the moments \( \{ \beta'_n(\lambda) \}_{n=0}^{\infty} \), higher precision determination of zeros of higher-degree Jensen polynomials, and more computer time! In this vein, we mention that the recent impressive results of van de Lune, te Riele and Winter [16], show that the first 1,500,000,001 zeros of the Riemann zeta function in the critical
strip are simple and do indeed satisfy \( \text{Re} z = 1/2 \). These computations involved about 900 CPU hours on a CYBER 175/750 and about 1000 CPU hours on a CYBER 235.

Because of their use in establishing our main result of (26), we consider in detail the moments \( \tilde{b}_m(\lambda) \) of (13), and their numerical approximation. Since each moment \( \tilde{b}_m(\lambda) \) of (13) can be expressed as the sum

\[
\tilde{b}_m(\lambda) = \int_0^1 t^{2m} e^{\lambda t^2} \Phi(t) \, dt + \int_1^\infty t^{2m} e^{\lambda t^2} \Phi(t) \, dt \quad (m = 0, 1, 2, \ldots),
\]

(27)

we first give an upper bound for the last integral of (27). Assuming that \( \lambda \) is negative, it is elementary to show that \( t^{2m} e^{\lambda t^2} \) is strictly decreasing for all \( t > 1 \), provided that \( 0 \leq m \leq |\lambda| \). Thus,

\[
\int_1^\infty t^{2m} e^{\lambda t^2} \Phi(t) \, dt < e^\lambda \int_1^\infty \Phi(t) \, dt < \frac{\pi}{2} \exp(\lambda + 5 - \pi e^4) \quad (0 \leq m \leq |\lambda|),
\]

(28)

the last inequality above being a consequence (cf. [4, Eq. (3.7)]) of the known inequality \( \int_1^\infty \Phi(t) \, dt < \frac{\pi}{2} \exp(5 - \pi e^4) \). On setting

\[
b_m^{(1)}(\lambda) = \int_0^1 t^{2m} e^{\lambda t^2} \Phi(t) \, dt \quad (m = 0, 1, \ldots),
\]

(29)

it follows from (28) and the positivity of the integrands of (27) that when \( \lambda = -50 \),

\[
b_m^{(1)}(-50) < \tilde{b}_m(-50) < b_m^{(1)}(-50) + 1.45 \cdot 10^{-94} \quad (0 \leq m \leq 50).
\]

(30)

Thus, the absolute error obtained by replacing \( \tilde{b}_m(-50) \) by \( b_m^{(1)}(-50) \) is at most \( 1.45 \cdot 10^{-94} \) for all \( 0 \leq m \leq 50 \).

Next, if \( \Phi_N(t) \) denotes the sum of the first \( N \) terms of (3), then it is known (cf. [4, Eq. (4.6)]) that

\[
0 < \Phi(t) - \Phi_N(t) < \pi N^3 \exp(5 t - \pi N^2 e^{4t}) \quad (0 \leq t < \infty),
\]

(31)

But for fixed \( N \geq 1 \), \( \exp(5 t - \pi N^2 e^{4t}) \) is strictly decreasing for \( t \geq 0 \), so that

\[
0 < \Phi(t) - \Phi_N(t) < \pi N^3 e^{-\pi N^2} \quad (0 \leq t < \infty),
\]

and on evaluating the above when \( N = 9 \), we have, in particular, that

\[
0 < \Phi(t) - \Phi_N(t) < 7.1 \cdot 10^{-108} \quad (0 \leq t \leq 1).
\]

(32)

On defining

\[
b_m^{(2)}(\lambda) = \int_0^1 t^{2m} e^{\lambda t^2} \Phi_N(t) \, dt \quad (0 \leq m \leq |\lambda|),
\]

(33)
we then have, in analogy with (30), that
\[
0 < b_{2m}^{(2)}(-50) < \beta_m(-50) < b_{2m}^{(2)}(-50) + 1.46 \cdot 10^{-94} \quad (0 \leq m \leq 50). \tag{34}
\]
For our problem at hand, the specific moments \(\{b_{2m}^{(2)}(-50)\}_{m=0}^{16}\) of (33) were numerically approximated, using the Romberg integration method.

Now, for each real \(\lambda\) and for each nonnegative integer \(m\), we note that the integrand of the integral of (33) is, from (3), in \(C^{2k+2}[0,1]\) for any positive integer \(k\). This smoothness of this integrand implies that the \((k+1)\)-st column \(T_{i,k}^{(m)}\) of the Romberg integration table, associated with the integral of (33), is convergent to the exact value of that integral when step lengths \(h_i = 2^{-i}\) \((i=0, 1, \ldots)\) are used (cf. Stoer and Bulirsch [14, Eq. (3.4.7), p. 135]). As this is true for each positive integer \(k\), then the extrapolation method of Romberg integration applied to the integral of (33), can produce arbitrarily accurate values of \(b_{2m}^{(2)}(\lambda)\).

For the Romberg method of integration applied to each of the moments \(\{b_{2m}^{(2)}(-50)\}_{m=0}^{16}\), all computations were performed in FORTRAN 77, using Brent’s MP package [1] for extended-precision floating-point numbers and 110 digits of precision on a VAX-11/780 in the Institute for Computational Mathematics at Kent State University. On denoting the entries of the Romberg table, associated with the integral \(b_{2m}^{(2)}(-50)\), by \(T_{i,k}^{(m)}(0 \leq m \leq 16)\), the actual iterations of the Romberg method (with step lengths \(h_i = 2^{-i}\)) were terminated in all cases when three successive entries, in a single column of \(T_{i,k}^{(m)}\), agreed to 65 significant digits in normalized floating-point representation. The final approximation to \(b_{2m}^{(2)}(-50)\), denoted by \(\bar{b}_m(-50)\), was then defined to be the largest values of \(k\) and \(i\) in the Romberg table such that \(T_{i,k}^{(m)}\) and \(T_{i+1,k}^{(m)}\) agreed to 65 significant digits in normalized floating-point representation. These computed moments \(\{\bar{b}_m(-50)\}_{m=0}^{16}\), truncated to 60 significant digits, are given in Table 1. In particular, this accuracy implies that
\[
\left| \frac{\bar{b}_m(-50)}{b_{2m}^{(2)}(-50)} - 1 \right| \leq 10^{-64} \quad (m = 0, 1, \ldots, 16),
\]
and thus, with (34) and the magnitude of the numbers \(\beta_m(-50)\) from Table 1, we have the conservative bounds
\[
\left| \frac{\bar{b}_m(-50)}{b_{2m}^{(2)}(-50)} - 1 \right| \leq 10^{-60} \quad (m = 0, 1, \ldots, 16). \tag{35}
\]
As an added precaution in estimating the moments \(\{b_{2m}^{(2)}(-50)\}_{m=0}^{16}\), we used a technique, described in Bulirsch and Stoer [3] and in [14, p. 141], for determining asymptotic upper and lower bounds for the numbers \(b_{2m}^{(2)}(-50)\). Specifically, from the entries \(T_{i,k}^{(m)}\) of the Romberg table associated with \(b_{2m}^{(2)}(-50)\), one can easily compute the numbers \(U_{i,k}^{(m)}\) defined by
\[
U_{i,k}^{(m)} = 2 T_{i+1,k}^{(m)} - T_{i,k}^{(m)},
\]
and these numbers \(U_{i,k}^{(m)}\) build a new associated Romberg-type table for the sought moment \(b_{2m}^{(2)}(-50)\). Because of the smoothness of the integrand of (33),
Table 1. The computed moments

<table>
<thead>
<tr>
<th>m</th>
<th>( \beta_m (\pm 50) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.19656 13339 29181 93310 73686 70252 18899 08075 14869 18579 39816 09600 (-2)</td>
</tr>
<tr>
<td>1</td>
<td>2.32011 71418 85536 82157 44972 45357 77375 47795 47788 78966 01105 0975 (-4)</td>
</tr>
<tr>
<td>2</td>
<td>3.77089 97677 30881 23738 16195 78125 66204 22778 11075 77741 16472 9297 (-6)</td>
</tr>
<tr>
<td>3</td>
<td>1.00227 69596 72457 01475 32275 01634 91380 18026 69849 21468 18998 4564 (-7)</td>
</tr>
<tr>
<td>4</td>
<td>3.66337 61735 70406 23425 94004 30741 73648 26291 73317 49864 18599 5080 (-9)</td>
</tr>
<tr>
<td>5</td>
<td>1.69252 58688 59480 57551 74685 29706 06315 55393 12811 30248 37182 8204 (-10)</td>
</tr>
<tr>
<td>6</td>
<td>9.40358 19102 26836 44833 49098 87957 05821 36909 15984 78893 65575 6861 (-12)</td>
</tr>
<tr>
<td>7</td>
<td>6.07928 26738 29840 63138 68635 44716 86713 97453 67318 45834 48369 0895 (-13)</td>
</tr>
<tr>
<td>8</td>
<td>4.46758 07993 72611 29511 35750 68116 37426 68930 24699 78052 39388 9235 (-14)</td>
</tr>
<tr>
<td>9</td>
<td>3.66774 22933 21191 60815 42487 62165 96955 97101 68171 08309 51339 1938 (-15)</td>
</tr>
<tr>
<td>10</td>
<td>3.31887 96848 26088 11013 54576 58364 37597 62937 04392 36885 19629 7525 (-16)</td>
</tr>
<tr>
<td>11</td>
<td>3.27495 66308 18925 42519 54660 46267 96438 12270 41617 07059 97861 9700 (-17)</td>
</tr>
<tr>
<td>12</td>
<td>3.49350 79889 03492 82136 66727 14088 12286 23184 09702 38293 93952 1310 (-18)</td>
</tr>
<tr>
<td>13</td>
<td>3.99970 36556 45639 31360 95058 35153 19042 31348 37182 39401 70449 4114 (-19)</td>
</tr>
<tr>
<td>14</td>
<td>4.88503 24854 26203 87448 03575 24930 26569 88256 82483 91503 16036 0221 (-20)</td>
</tr>
<tr>
<td>15</td>
<td>6.33188 05610 98341 63013 78232 19621 78670 63968 85444 40652 31370 0822 (-21)</td>
</tr>
<tr>
<td>16</td>
<td>8.67132 23144 43954 11051 39659 30824 92230 12635 52164 34153 16708 6153 (-22)</td>
</tr>
</tbody>
</table>

then for each nonnegative integer \( k \), the first column (corresponding to the trapezoidal rule) of the Romberg table for \( b_m^{(2)} (-50) \), can be written symbolically as

\[
T_h = b_m^{(2)} (-50) + \tau_1 h^2 + \tau_2 h^3 + \ldots + \tau_k h^{2k} + (\tau_{k+1} + o(1)) h^{2k-2},
\]

where the \( \tau_k \) are independent of \( h \). If \( \tau_{k+1} \neq 0 \), it is shown in [3] that the \((k+1)\)-st columns \( T_{h_{k+1}}^{(m)} \) and \( U_{h_{k+1}}^{(m)} \) of these tables eventually converge monotonically to \( b_m^{(2)} (-50) \), as \( h \to \infty \), at asymptotically the same rate, but from opposite directions. Hence, these numbers thus provide asymptotically upper and lower bounds for \( b_m^{(2)} (-50) \). Indeed, in all cases of our computations, the final elements of the first columns of \( T_{h_{k+1}}^{(m)} \) and \( U_{h_{k+1}}^{(m)} \) were respectively monotone increasing, and monotone decreasing, i.e.,

\[
T_{h_{0}}^{(m)} < T_{h_{0}}^{(m)} < T_{h_{0}}^{(m)} \quad \text{and} \quad U_{h_{0}}^{(m)} > U_{h_{0}}^{(m)} \quad (m = 0, 1, \ldots, 16).
\]

This monotonicity suggests that each interval \([T_{h_{0}}^{(m)}, U_{h_{0}}^{(m)}]\) contains \( b_m^{(2)} (-50) \), for all \( m = 0, 1, \ldots, 16 \).

While it might be interesting for the reader to peruse the bulk of digits of the tables \( T_{h_{k}}^{(m)} \) and \( U_{h_{k}}^{(m)} (m = 0, 1, \ldots, 16) \), we have opted instead to give in Table 2 the much briefer listing of the number of significant digits to which \( T_{h_{k}}^{(m)} \) and \( U_{h_{k}}^{(m)} \) actually agree. From Table 2, we note two interesting items. First, we note that the agreement between \( T_{h_{k}}^{(m)} \) and \( U_{h_{k}}^{(m)} \) in Table 2 start in the case \( m = 0 \) with the surprisingly high number 93, and second, that this agreement decays to 73 digits when \( m = 15 \). But, this 73 digits of agreement is still very conservative for what is claimed in (35).
Table 2. Significant digits of agreement between \( T_{\alpha, 0} \) and \( U_{\alpha, 0} \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>Significant digits of agreement</th>
<th>( m )</th>
<th>Significant digits of agreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>93</td>
<td>9</td>
<td>79</td>
</tr>
<tr>
<td>1</td>
<td>91</td>
<td>10</td>
<td>79</td>
</tr>
<tr>
<td>2</td>
<td>89</td>
<td>11</td>
<td>78</td>
</tr>
<tr>
<td>3</td>
<td>88</td>
<td>12</td>
<td>76</td>
</tr>
<tr>
<td>4</td>
<td>85</td>
<td>13</td>
<td>75</td>
</tr>
<tr>
<td>5</td>
<td>85</td>
<td>14</td>
<td>74</td>
</tr>
<tr>
<td>6</td>
<td>82</td>
<td>15</td>
<td>74</td>
</tr>
<tr>
<td>7</td>
<td>82</td>
<td>16</td>
<td>73</td>
</tr>
<tr>
<td>8</td>
<td>81</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof of the Theorem. From (21), set \( G_{16}(t; -50) := \sum_{j=0}^{16} b_j t^j \), and from (23), set \( g_{16}(t; -50) := \sum_{j=0}^{16} a_j t^j \). From (35), it follows that (24) is satisfied with \( c = 10^{-60} \), i.e.,

\[
|b_j - a_j| \leq 10^{-60} |a_j| \quad (0 \leq j \leq 16).
\]

Thus, from (25) of Proposition 2, the zeros \( \{w_j\}_{j=1}^{16} \) of \( G_{16}(t; -50) \) and the zeros \( \{z_j\}_{j=1}^{16} \) of \( g_{16}(t; -50) \) are coupled through

\[
\left| \frac{w_j}{z_j} - 1 \right| < 8 \sqrt{t^{1/n}} = 8 \cdot 16 (10^{-60})^{1/16} < 0.022762.
\]

(37)

We remark that the Jenkins algorithm [4], implemented in MP with 110 digits of precision, was used to determine all zeros of the polynomial \( g_{16}(t; -50) \). (The zeros of \( g_{16}(t; -50) \), truncated to twenty decimal digits, all given in Table 3.) From Table 3, we see that \( g_{16}(t; -50) \) has a complex zero

\[
z_2 := -220.91911 17368 44952... + i 7.09256 52553 63885 67...
\]

(38)

whose modulus is

\[
|z_2| = 221.03293 51307 13450...
\]

On multiplying by \( |z_2| \) in (37) for the case \( j = 2 \), we obtain

\[
|w_2 - z_2| < 0.022762 \cdot |z_2| < 5.03116.
\]

(39)

It is evident from (38) and (39) that

\[
\text{Im } w_2 > 2.0614,
\]

(40)

whence \( G_{16}(t; -50) \) has a nonreal zero. Applying Proposition 1 then gives the desired result of (26). □

As a final remark, the specific numbers \(-50\) (as a lower bound for \( A \)) and \( 16 \) (as the degree of the associated Jensen polynomial \( G_{16}(t; -50) \) having some
Table 3. The computed zeros of \( g_{16}(t; -50) \)

<table>
<thead>
<tr>
<th>Real part</th>
<th>Imaginary part</th>
</tr>
</thead>
<tbody>
<tr>
<td>-138.94239 25728 27322 76</td>
<td>-</td>
</tr>
<tr>
<td>-220.91911 17368 44952 75</td>
<td>7.09256 52553 63889 6761</td>
</tr>
<tr>
<td>-220.91911 17368 44952 75</td>
<td>-7.09256 52553 63889 6761</td>
</tr>
<tr>
<td>-286.00232 55522 2961 55</td>
<td>-</td>
</tr>
<tr>
<td>-310.79782 67793 62579 98</td>
<td>-</td>
</tr>
<tr>
<td>-350.18851 27945 36537 80</td>
<td>-</td>
</tr>
<tr>
<td>-386.00014 13750 66719 39</td>
<td>-</td>
</tr>
<tr>
<td>-422.33256 42144 23880 51</td>
<td>-</td>
</tr>
<tr>
<td>-459.76615 10040 07194 18</td>
<td>-</td>
</tr>
<tr>
<td>-498.36874 57252 89981 53</td>
<td>-</td>
</tr>
<tr>
<td>-538.57501 58487 17219 65</td>
<td>-</td>
</tr>
<tr>
<td>-580.89029 00913 24826 13</td>
<td>-</td>
</tr>
<tr>
<td>-626.04149 21507 01991 05</td>
<td>-</td>
</tr>
<tr>
<td>-675.20466 35826 2683 01</td>
<td>-</td>
</tr>
<tr>
<td>-730.62228 63983 20152 69</td>
<td>-</td>
</tr>
<tr>
<td>-798.10604 06173 97392 18</td>
<td>-</td>
</tr>
</tbody>
</table>

Nonreal zeros) appear somewhat magically in this section. To explain this further, our first high-precision attempt to constructively determine a lower bound for \( A \) began with \( \lambda = -100 \), and the associated approximate Jensen polynomials of (23), i.e., \( \{g_n(t; -100)\}_{n=1}^{15} \), all had only real negative zeros, while \( g_{10}(t; -100) \) possessed a nonreal zero. Now, the precision of these calculations were much more than adequate, in applying Proposition 2, to deduce the lower bound

\[-100 < A.\]

As this was almost too easy, the process was repeated with \( \lambda = -50 \). The associated approximate Jensen polynomials \( \{g_n(t; -50)\}_{n=1}^{15} \) had only real negative zeros, while \( g_{16}(t; -50) \) possessed a nonreal zero. This time, however, the precision of these calculations were just within the applicability of Proposition 2 to deduce that \( G_{16}(t; -50) \) has a nonreal zero, whence

\[-50 < A.\]

These calculations support our optimism that further improved lower bounds for \( A \) are possible!

Appendix A

Our object here is to show that the entire function (cf. (9)) \( H_\lambda(x) \), defined by

\[ H_\lambda(x) := \frac{1}{2} \int_{-\infty}^{+\infty} e^{ixt} \Phi(t) e^{ixt} dt = \int_{0}^{\infty} e^{ixt} \Phi(t) \cos(xt) dt, \]

(A1)
is of order 1, and of maximal type, (i.e., its type, \( \sigma_\lambda \), satisfies \( \sigma_\lambda = \infty \), for all \( \lambda \in \mathbb{R} \). We remark that this is known to be true for the special case \( \lambda = 0 \), since the Riemann \( \xi \)-function is of order one and of maximal type (cf. Titchmarsh [15, p. 29]) and since \( H_0(x) = \xi(x/2)/8 \). For other values of real \( \lambda \), this may be a known result, but we are unaware of any references for this. We do know that related results can be derived from the fact that the following differential identity holds, namely

\[
H_\lambda(x) = F_\lambda(D) H_0(x), \quad \left( D = \frac{d}{dx} \right)
\]

where

\[
F_\lambda(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m z^{2m}}{m!}.
\]

Such results (cf. Sikkema [13, p. 100] and van der Steen [17, p. 62]) give the weaker results that \( H_\lambda(x) \) is either of order 1 and type at most \( \infty \), or of order less than 1, for any \( \lambda \in \mathbb{R} \). In contrast with the above references, our technique of proof uses asymptotic series which may have independent interest.

To begin, for any \( \lambda \in \mathbb{R} \), set \( M_\lambda(r) := \max_{|z|=r} |H_\lambda(z)| \), so that \( M_\lambda(r) \) is the maximum modulus function for \( H_\lambda(z) \). It an be verified from (A1) that

\[
M_\lambda(r) := \int_0^\infty e^{\lambda t^2} \Phi(t) \cosh(rt) \, dt,
\]

so that

\[
\frac{1}{2} \int_0^\infty e^{\lambda t^2} \Phi(t) e^{rt} \, dt \leq M_\lambda(r) \leq \int_0^\infty e^{\lambda t^2} \Phi(t) e^{rt} \, dt,
\]

(A2)

for all \( r \geq 0 \) and any \( \lambda \in \mathbb{R} \). Next, from [4, eq. 3.41 and Thm. A(i)], we have

\[
\pi (2 \pi e^{4t} - 3) \exp(5 t - \pi e^{4t}) < \Phi(t) < \frac{203}{202} \pi (2 \pi e^{4t} - 3) \exp(5 t - \pi e^{4t}),
\]

for all \( t \geq 0 \), and as \( (2 \pi - 3) e^{4t} < 2 \pi e^{4t} - 3 < 2 \pi e^{4t} \) for all \( t > 0 \), then

\[
\frac{\pi}{2} (2 \pi - 3) \int_0^\infty \exp[(9 + r) t + \lambda t^2 - \pi e^{4t}] \, dt < M_\lambda(r) < \frac{203}{202} \frac{\pi}{2} \int_0^\infty \exp[(9 + r) t + \lambda t^2 - \pi e^{4t}] \, dt
\]

(A3)

for all \( r \geq 0 \). Thus, we set

\[
I(r, \lambda) := \int_0^\infty \exp[(9 + r) t + \lambda t^2 - \pi e^{4t}] \, dt = \int_0^\infty e^{h(t)} \, dt,
\]

(A4)

where

\[
h(t) = h(t; r, \lambda) = (9 + r) t + \lambda t^2 - \pi e^{4t} (t \geq 0),
\]

(A5)
We have

\[ h'(t) = 9 + r + 2 \lambda t - 4 \pi e^{4t}, \]
\[ h''(t) = 2 \lambda - 16 \pi e^{4t}, \]
\[ h^{(m)}(t) = -4^m \pi e^{4t} (m \geq 3). \]  

(A 6)

In what follows, \( \lambda \) is a fixed real number, and \( r(\geq 0) \) is a large parameter, tending to \( +\infty \). Since our assertion that \( H_2(x) \) is of order 1 and maximal type is known to be true for \( \lambda = 0 \), we may assume \( \lambda \neq 0 \).

**Lemma 1.** Given any \( \lambda \in \mathbb{R} \), then for each \( r > 0 \) sufficiently large, there is a unique \( \hat{t} = \hat{t}(r, \lambda) \) for which \( h'(\hat{t}) = 0 \).

**Proof.** Since \( h^{(3)}(t) = -64 \pi e^{4t} < 0 \) all \( t \geq 0 \), then \( h''(t) \) is strictly decreasing for \( t \geq 0 \), with \( h''(t) \rightarrow -\infty \) as \( t \rightarrow +\infty \). Next, \( h''(0) = 2 \lambda - 16 \pi \). If \( \lambda \leq 8 \pi \), set \( t = 0 \). If \( \lambda > 8 \pi \), let \( \tilde{t} > 0 \) be the unique value of \( t > 0 \) such that \( h''(\tilde{t}) = 0 \). Thus, in all cases, \( h''(\tilde{t}) < 0 \) for all \( t > \tilde{t} \). Then, choose \( r > 0 \) sufficiently large so that \( h'(\tilde{t}) > 0 \). Because \( h'(t) \) is strictly decreasing (for \( t > \tilde{t} \)), with \( h'(\tilde{t}) > 0 \) and with \( h'(t) \rightarrow -\infty \) as \( t \rightarrow \infty \), there is evidently a unique \( \hat{t} > \tilde{t} \) for which \( h'(\hat{t}) = 0 \).

By construction, we note that \( h'(\hat{t}) < 0 \) for all \( r \) sufficiently large, so that the Taylor expansion of \( h(t) \), about \( \hat{t} \), can be written as

\[ h(t) = h(\hat{t}) - \frac{|h''(\hat{t})|}{2} (t - \hat{t})^2 + \sum_{m=3}^{\infty} \frac{h^{(m)}(\hat{t})}{m!} (t - \hat{t})^m. \]  

(A 7)

We now estimate \( \hat{t} = \hat{t}(r, \lambda) \). By definition, \( h'(\hat{t}) = 0 \) implies from (A 6) that

\[ 4\pi e^{4\hat{t}} = 9 + r + 2 \lambda \hat{t} \]

Calling \( 4\hat{t} = \tilde{\alpha}, (9 + r)/4\pi = \beta \), and \( \lambda/8\pi = \alpha \), the above equation becomes

\[ \tilde{\alpha} = \beta + \alpha \tilde{u}. \]  

(A 8)

**Lemma 2.** For \( \beta > 0 \) sufficiently large, the unique solution \( \tilde{u} \) of (A 8) satisfies

\[ \log[\beta + \alpha \log \beta] < \tilde{u} < \log \beta + \frac{\alpha \log \beta}{\beta}. \]  

(A 9)

Thus

\[ \hat{t} = \frac{1}{4} \left\{ \log \beta + \frac{\alpha \log \beta}{\beta} + O\left(\left(\frac{\log \beta}{\beta}\right)^2\right) \right\}, \text{ as } \beta \rightarrow \infty. \]  

(A 10)

**Proof.** From (A 6) and (A 8), we have that \( h'(t)/4\pi = \{-e^\beta + \beta + \alpha u\} = g(u) \). If we set \( \tilde{u} = \log \beta + \alpha \log \beta/\beta \), then a calculation shows that

\[ g(\tilde{u}) = -\frac{\alpha^2 \log^2 \beta}{2\beta} \left( \frac{1}{\log \beta} - \frac{2}{\beta} \right) + O\left( \frac{\alpha^3 \log^3 \beta}{\beta^2} \right), \text{ as } \beta \rightarrow +\infty. \]

Thus, since \( \lambda \neq 0 \) implies \( \alpha \neq 0 \), then for \( \beta > 0 \) sufficiently large, we have \( g(\tilde{u}) < 0 \), whence (from the monotonicity of Lemma 1) \( \hat{u} < \tilde{u} \). Similarly, with \( u = \alpha \alpha \log \beta \).
\[ \log(\beta + x \log \beta), \] a calculation shows that

\[
g(u) = \frac{\log \left( 1 + \frac{x \log \beta}{\beta} \right)}{\beta} = \frac{x^2 \log \beta}{\beta} \left( 1 - \frac{1}{2} \left( \frac{x \log \beta}{\beta} \right) + O \left( \left( \frac{x \log \beta}{\beta} \right)^2 \right) \right) > 0,
\]

for all \( \beta \) sufficiently large, whence \( g(u) > 0 \). But as \( g(\hat{u}) = 0 \), the monotonicity of Lemma 1 gives that \( y < \hat{u} < \bar{u} \) for all \( \beta \) sufficiently large, which is (A9). Finally, \( u := \log(\beta + x \log \beta) = \log \beta + \log \left( 1 + \frac{x \log \beta}{\beta} \right) = \log \beta + \frac{x \log \beta}{\beta} + O \left( \left( \frac{x \log \beta}{\beta} \right)^2 \right), \) \( \beta \to \infty \). This means that \( \bar{u} - y = O \left( \left( \frac{x \log \beta}{\beta} \right)^2 \right) \) and as \( \beta = \lambda/8\pi \) is a fixed nonzero constant, then \( \bar{u} = 4\hat{\gamma} \) evidently satisfies (A10).

We now estimate the terms \( \{h^{(m)}(\hat{t})\}_{m=0}^\infty \) of (A7). A short calculation shows that

\[ h(\hat{t}) = \pi \beta \log \beta \left( 1 - \frac{1}{\log \beta} + O \left( \frac{\log \beta}{\beta} \right) \right), \]

\[ h'(\hat{t}) = -16\pi \beta \left( 1 + O \left( \frac{\log \beta}{\beta} \right) \right), \]

\[ h^{(m)}(\hat{t}) = -4^n \pi \beta \left( 1 + O \left( \frac{\log \beta}{\beta} \right) \right), (m \geq 3), \beta \to +\infty. \] \hfill (A11)

Next, from (A7) and (A4), we have that

\[ I(r, \lambda) = e^{h(\hat{t})} \int_0^\infty \exp \left\{ -\frac{|h'(\hat{t})|}{2} (t - \hat{t})^2 + \sum_{m=3}^\infty \frac{h^{(m)}(\hat{t})}{m!} (t - \hat{t})^m \right\} dt. \]

Setting \( t - \hat{t} = v \), then

\[ I(r, \lambda) = e^{h(\hat{t})} \int_{-\infty}^{\infty} e^{-\frac{|h'(\hat{t})|}{2} v^2} e^{Q(v)} dv, \]

where \( Q(v) := \sum_{m=3}^\infty h^{(m)}(\hat{t}) v^m/m! \). Setting

\[ e^{Q(v)} := 1 + \sum_{j=3}^\infty \mu_j v^j, \text{ where } \mu_j = \frac{h^{(m)}(\hat{t})}{m!} \text{ for } j = 3, 4, \text{ and } 5, \]

then we have, since \( \hat{t} \to +\infty \) as \( r \to \infty \), that (in the sense of asymptotic series, cf. Henrici [5, Ch. 11])

\[ I(r, \lambda) = e^{h(\hat{t})} \int_{-\infty}^{\infty} e^{-\frac{|h'(\hat{t})|}{2} v^2} \left( 1 + \mu_3 v^3 + \mu_4 v^4 + \ldots \right) dv. \]
\[ e^k(i) \left\{ \int_{-\infty}^{+\infty} e^{\frac{-|u|}{2}} du + \mu_4 \int_{-\infty}^{+\infty} e^{\frac{-|u|}{2}} v^4 du + \ldots \right\}, \]

\[ = e^{k(i)} \left\{ \sqrt{\frac{2\pi}{|h''(\hat{t})|}} + \frac{h^{(4)}(\hat{t})}{8(h''(\hat{t}))^2} \sqrt{\frac{2\pi}{|h''(t)|}} + \ldots \right\} \]

\[ = e^{k(i)} \sqrt{\frac{2\pi}{|h''(\hat{t})|}} \left\{ 1 + \frac{h^{(4)}(\hat{t})}{8(h''(\hat{t}))^2} + \ldots \right\} = e^{k(i)} \sqrt{\frac{2\pi}{|h''(\hat{t})|}} \left\{ 1 + O\left(\frac{1}{\beta}\right) \right\}. \]

Using the results of (A11) in the above, we deduce, after some calculations, that

\[ I(r, \lambda) = e^{\pi \beta \log \beta - \beta} + O(\log \beta), \quad r \to \infty. \]  

(A12)

Thus, upon recalling that \( \beta = (9 + r)/4\pi \), this gives that

\[ \log I(r, \lambda) = \frac{r \log r}{4} + O(r), \]

(A13)

as \( r \to \infty \). From this, there follows

\[ \log \log I(r, \lambda) = \log r + \log \log r + O(1), \quad r \to \infty, \]  

(A14)

so that

\[ \lim_{r \to \infty} \frac{\log \log I(r, \lambda)}{\log r} = 1. \]  

(A15)

But since we have from (A3) that

\[ \frac{\pi}{2} (2\pi - 3) I(r, \lambda) \leq M_{\lambda}(r) \leq \left(\frac{203}{202}\right) 2\pi^2 I(r, \lambda) \quad (\forall r \geq 0), \]

then on taking logarithms and using (A13), we have

\[ \lim_{r \to \infty} \frac{\log M_{\lambda}(r)}{r \log r} = 4. \]  

(A16)

Hence, from (A16), we then have that

\[ \lim_{r \to \infty} \frac{\log \log M_{\lambda}(r)}{\log r} = 1 \quad (\forall \lambda \in \mathbb{R}). \]  

(A17)

This proves that \( H_{\lambda}(z) \) is an entire function of order 1 for any \( \lambda \in \mathbb{R} \). Next, as \( H_{\lambda}(z) \) is of order 1, its type, \( \sigma_{\lambda} \), is defined as usual by

\[ \lim_{r \to \infty} \frac{\log M_{\lambda}(r)}{r} =: \sigma_{\lambda}. \]

But from (A16), it is evident that \( \sigma_{\lambda} = +\infty \) for all \( \lambda \in \mathbb{R} \).
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References


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Errata for
“A Lower Bound for the de Bruijn-Newman Constant Λ”,
by G. Csordas, T.S. Norfolk, and R.S. Varga

The following corrections should be made:

- p. 486, eq. 20. Read “∑_{m=0}^{∞}” for “∑_{j=0}^{∞}”.
- p. 493, line +16. Read “It can be...” for “It an be...”.
- p. 494, line +15. Read “a unique \hat{t} > \tilde{t}...” for “a unique \hat{t} > \tilde{t}...”.
- p. 494, line +17. Read “about \hat{t}...” for “about \tilde{t}...”.
- p. 496, line +2. At the end of this display, read “√{2π \over \ln^2(t)}” for “√{2π \over \ln^2(t)}.”