On a 2-Periodic Lacunary Trigonometric Interpolation Problem

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Abstract. In this paper we obtain simple necessary and sufficient conditions for a particular 2-periodic lacunary trigonometric interpolation problem on equidistant nodes in \([0, 2\pi]\) to be regular.

§1. Introduction

For a given positive integer \(n\), define

\[
x_k = x_k(n) := k\pi / n \quad (k = 0, 1, \ldots, 2n - 1),
\]

so that \(\{x_k\}_{k=0}^{2n-1}\) is a set of 2n equidistant nodes in \([0, 2\pi]\). Next, assume that

\[
\{m_j\}_{j=1}^p \text{ are any } p \text{ distinct positive integers } (p \geq 1).
\]

We consider here the following 2-periodic lacunary trigonometric interpolation problem, denoted by the expression

\[
(0 =: m_0, m_1, \ldots, m_p; m_1, m_2, \ldots, m_p),
\]

on the 2n equidistant nodes \(\{x_k\}_{k=0}^{2n-1}\) in \([0, 2\pi]\). For arbitrary data consisting of complex numbers \(\{\alpha_{j, \nu}\}_{j=0, \nu=0}^{n-1, m_p}\) and \(\{\beta_{j, \nu}\}_{j=0, \nu=1}^{n-1, m_p}\), we ask if there is a unique trigonometric polynomial of the form

\[
t_M(x) = a_0 + \sum_{k=1}^{M} (a_k \cos kx + b_k \sin kx),
\]

or of the form

\[
t_M(x) = a_0 + \sum_{k=1}^{M-1} (a_k \cos kx + b_k \sin kx) + a_M \cos \left(Mx + \frac{\pi}{2}\right),
\]

(1.4)
(where $\varepsilon = 0$ or where $\varepsilon = 1$), such that

\[
\begin{aligned}
\ell^{(m_{\nu})}_M(x_{2j}) &= \alpha_{j,\nu} \quad (j = 0, 1, \ldots, n - 1; \quad \nu = 0, 1, \ldots, p), \\
\ell^{(m_{\nu})}_M(x_{2j+1}) &= \beta_{j,\nu} \quad (j = 0, 1, \ldots, n - 1; \quad \nu = 1, 2, \ldots, p).
\end{aligned}
\]

(1.5)

Note that as the number of nodes in (1.1) is even, we see that the interpolation conditions of (1.5) break down into interpolation conditions on the disjoint sets $\{x_{2j}\}_{j=0}^{n-1}$ and $\{x_{2j+1}\}_{j=0}^{n-1}$ of nodes, each set consisting of $n$ nodes. From this, the term 2-periodic lacunary trigonometric interpolation is derived. In addition, we see from (1.5) that the first group of integers of (1.3) give the derivative conditions on the set $\{x_{2j}\}_{j=0}^{n-1}$, while the latter group of integers of (1.3) give the derivative conditions on $\{x_{2j+1}\}_{j=0}^{n-1}$.

It is evident that the total number of interpolation conditions in (1.5) is

\[
N := n(2p + 1).
\]

(1.6)

Then, as $N$ is odd iff $n$ is odd, the desired trigonometric interpolant $t_M(x)$ in (1.5) when $N$ is odd is necessarily of the form (1.4) (which has an odd number of parameters), and $M = (N - 1)/2$ in this case. Continuing, as $N$ is even iff $n$ is even, the desired trigonometric polynomial $t_M(x)$ in (1.5) when $N$ is even is necessarily of the form (1.4') with $M = N/2$, where $\varepsilon = 0$ or 1 is to be determined. We say that this $\ell (0, m_1, \ldots, m_p; m_1, \ldots, m_p)$ 2-periodic lacunary interpolation problem is regular if, for arbitrary data, (1.5) admit a unique solution, where $t_M(x)$ is of the form (1.4) when $N$ is odd, or of the form (1.4') when $N$ is even.

The goal of this paper is to derive simple (i.e., non-determinantal) necessary and sufficient conditions on $N, n$, the integers $\{m_j\}_{j=1}^p$, and $\varepsilon$ (when (1.4') is used) for the 2-periodic lacunary trigonometric interpolation problem (1.3) to be regular. As we shall see below, this goal is reached.

§2. Main Result

For notation, let

\[
\begin{aligned}
q &= \text{number of even integers in the set } \{m_j\}_{j=1}^p \text{ of (1.2)}, \\
\sigma &= \text{number of odd integers in the set } \{m_j\}_{j=1}^p \text{ of (1.2)},
\end{aligned}
\]

(2.1)

so that

\[
e_p + o_p = p.
\]

(2.2)

Our main result is the following
Theorem 1. Let \( \{m_j\}_{j=1}^p \) be \( p \) distinct positive integers and let \( \{x_k(n)\}_{k=0}^{2n-1} \) be the \( 2n \) equidistant nodes in \([0, 2\pi]\) of (1.1).

1. If \( N := n(2p+1) \) is odd, so that \( n \) is also odd, then the 2-periodic trigonometric interpolation problem (1.5), with \( t_M(x) \) of the form (1.4), is regular iff \( p \) is even and (cf. (2.1))

\[
e_p = o_p = p/2.
\]

2. If \( N := n(2p+1) \) is even, so that \( n \) is also even, then the 2-periodic trigonometric interpolation problem (1.5), with \( t_M(x) \) of the form (1.4'), is regular iff \( p \) is even, (2.3) is satisfied, and \( \varepsilon = 0 \) in (1.4').

In particular, the interpolation problem (1.5) is never regular when \( p \) is odd.

Proof. (Sketch for \( N \) odd): Assume that \( N \) is odd, so that \( n \) is also odd from (1.6), and we write \( n = 2r + 1 \). In this case, \( M = (N - 1)/2 = np + r \), and the desired trigonometric polynomials \( t_M(x) \) are of the form (1.4). Using the familiar device for identifying trigonometric polynomials with algebraic polynomials through the transformation \( z = e^{ix} \), any trigonometric polynomial \( t_M(x) \) of the form (1.4) can be expressed as

\[
t_M(x) = z^{-M} q_{2M}(z),
\]

where \( q_{2M}(z) \) is a complex polynomial of degree at most \( 2M \). On considering null data in (1.5) (i.e., \( \alpha_{j,\nu} = 0 = \beta_{j,\nu} \) in (1.5)), \( t_M(x) \) can then be expressed as

\[
t_M(x) = z^{-M}(z^n - 1) \sum_{\lambda=0}^{2p-1} z^{\lambda n} \sum_{j=0}^{n-1} a_{\lambda,j} z^j \quad (n = 2r + 1),
\]

where the \( 2p \) unknowns \( \{a_{\lambda,j}\}_{\lambda=0}^{2p-1} \) can be shown to satisfy (for each \( j = 0, 1, \ldots, n - 1 \)) the \( 2p \) homogeneous equations:

\[
\sum_{\lambda=0}^{2p-1} a_{\lambda,j} \{ (\alpha_j + \lambda + 1 - p)^{m_{\nu}} - (\alpha_j + \lambda - p)^{m_{\nu}} \} = 0 \quad (\nu = 1, 2, \ldots, p),
\]

\[
\sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,j} \{ (\alpha_j + \lambda + 1 - p)^{m_{\nu}} + (\alpha_j + \lambda - p)^{m_{\nu}} \} = 0 \quad (\nu = 1, 2, \ldots, p).
\]

(2.6)

where (since \( n = 2r + 1 \))

\[
\alpha_j := (j - r)/(2r + 1) \quad (j = 0, 1, \ldots, 2r).
\]

(2.7)

Thus, if \( \Delta(\alpha_j) \) denotes the determinant of order \( 2p \) of the coefficients of \( \{a_{\lambda,j}\}_{\lambda=0}^{2p-1} \), then \( \Delta(\alpha_j) \neq 0 \) (for all \( j = 0, 1, \ldots, 2r \)) iff \( t_M(x) \equiv 0 \) in (2.4).
In other words, for the interpolation problem (1.5) to be regular in the case when $N$ is odd, it is necessary and sufficient that

$$
\Delta(\alpha_j) \neq 0 \quad (\alpha_j := (j - r)/(2r + 1); \quad j = 0, 1, \cdots, 2r).
$$ (2.8)

Since $\alpha_j = 0$ when $j = r$, a close examination of the particular determinant $\Delta(0)$, arising from (2.6) in the case $j = r$, shows that $\Delta(0) \neq 0$ implies that $p$ is even and that (cf. (2.1)) $e_p = o_p = p/2$. In other words, necessary conditions that the interpolation problem (1.5) be regular in this case when $N$ is odd are that

$$p \text{ is even, and } e_p = o_p = p/2.
$$ (2.9)

Conversely, in the case when $N$ is odd, a lengthy proof, using determinantal tools, shows that (2.9) implies that $\Delta(\alpha_j) \neq 0$ for all $j = 0, 1, \cdots, 2r$, which, from (2.8), is necessary and sufficient for regularity.

The proof when $N$ is even is similar but more involved, as it requires, from (1.4'), the additional determination of $\varepsilon = 0$ or $\varepsilon = 1$. ■

References


Research supported in part by the Air Force Office of Scientific Research and by the National Science Foundation.