SOME 2-PERIODIC TRIGONOMETRIC INTERPOLATION PROBLEMS ON EQUIDISTANT NODES

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ABSTRACT: The problem to be considered here is the determination of necessary and sufficient conditions for the uniqueness of 2-periodic lacunary trigonometric interpolation on equidistant nodes. Our main results are new necessary and sufficient conditions in particular cases which depend only on the total number of even and odd integers in the derivatives which define the trigonometric interpolation process. As such, these new conditions can be readily checked, as they avoid the evaluation of determinants.

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1. Introduction.

Given any positive integer \( n \), define

\[
(1.1) \quad x_k = x_k(n) := \frac{k\pi}{n} \quad (k = 0, 1, \ldots, 2n - 1),
\]

so that \( \{x_k\}_{k=0}^{2n-1} \) is a set of \( 2n \) equidistant nodes in \([0, 2\pi)\). Next, with \( p \) and \( q \) arbitrary

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positive integers, assume that the associated vectors
\begin{equation}
\mathbf{m} := (m_1, m_2, \cdots, m_p), \text{ and } \mathbf{m'} := (m'_1, m'_2, \cdots, m'_q),
\end{equation}
have nonnegative integer components which satisfy
\begin{equation}
0 =: m_1 < m_2 < \cdots < m_p, \text{ and } 0 \leq m'_1 < m'_2 < \cdots < m'_q.
\end{equation}
We consider here the following 2-periodic trigonometric interpolation problem for the vectors \((\mathbf{m}; \mathbf{m'})\). For arbitrary data consisting of complex numbers \(\{\alpha_{j,\nu}\}_{j=0, \nu=1}^{n-1, p}\) and \(\{\alpha'_{j,\nu}\}_{j=0, \nu=1}^{n-1, q}\), we ask if there is a unique complex trigonometric polynomial of the form
\begin{equation}
t_M(x) = a_0 + \sum_{k=1}^{M} (a_k \cos kx + b_k \sin kx),
\end{equation}
or of the form
\begin{equation}
t'_M(x) = a_0 + \sum_{k=1}^{M-1} (a_k \cos kx + b_k \sin kx) + a_M \cos(Mx + \frac{\varepsilon \pi}{2})
\end{equation}
(where \(\varepsilon = 0\) or \(\varepsilon = 1\), such that
\begin{equation}
\begin{cases}
t_M^{(m)}(x_{2j}) = \alpha_{j,\nu} & (j = 0, 1, \cdots, n-1; \nu = 1, 2, \cdots, p), \text{ and} \\
t_M^{(m')} (x_{2j+1}) = \alpha'_{j,\nu} & (j = 0, 1, \cdots, n-1; \nu = 1, 2, \cdots, q).
\end{cases}
\end{equation}
Note that as the number of nodes in (1.1) is even (namely, 2n), we see that the interpolation conditions of (1.5) are broken into interpolation conditions on two disjoint sets of n nodes, from which the term, 2-periodic trigonometric interpolation, is derived.

The total number of interpolation conditions in (1.5) is evidently
\begin{equation}
N := n(p + q).
\end{equation}
If \(N\) is odd (which implies that \(n\) is odd and that \(p + q\) is odd), then the sought trigonometric interpolant \(t_M(x)\) is necessarily of the form (1.4) (which has an odd number of parameters), and in this case, \(M = (N-1)/2\). If \(N\) is even (which implies that at least one of the numbers, \(n\) and \(p + q\), is even), the sought trigonometric interpolant \(t_M(x)\) is necessarily of the form
(1.4') with \( M = N/2 \), where \( \epsilon (= 0 \text{ or } 1) \) is to be appropriately determined. To summarize,

\[
(1.7) \quad \begin{cases} 
M = (N - 1)/2 & \text{if } N \text{ is odd, and} \\
M = N/2 & \text{if } N \text{ is even.}
\end{cases}
\]

We say that this \((m; m')\) 2-periodic trigonometric interpolation problem of (1.5) is regular, if, for arbitrary data, the interpolation problem (1.5) admits a unique solution \( t_M(x) \), where \( t_M(x) \) is of the appropriate form (1.4) or (1.4').

This \((m; m')\) 2-periodic trigonometric interpolation problem on \( 2n \) equidistant nodes is a special case of the more general \( s \)-periodic trigonometric interpolation problem on \( sn \) equidistant nodes in \([0, 2\pi)\), considered in Sharma, Smith, and Tzimbalario [5]. In [5], necessary and sufficient conditions for the regularity of this \( s \)-periodic trigonometric interpolation problem were derived, in terms of the nonvanishing of several determinants (of possibly large order). In this form, these necessary and sufficient conditions are not in general easy to apply.

Recently, however, two papers (cf. Sharma, Szabados, and Varga [6] and Sharma and Varga [7]) have treated special cases of 2-periodic trigonometric interpolation on equidistant nodes, and, in each of these latter two papers, new necessary and sufficient conditions for regularity were derived which depend only on the total number of even and odd integers in the components of the vectors \( m \) and \( m' \) of (1.2). As such, these necessary and sufficient conditions can be easily checked, in contrast with the necessary and sufficient determinantal conditions of [5]. Thus, our goal here is to extend the results of [6] and [7] by finding necessary and sufficient conditions, for the regularity of the 2-periodic lacunary trigonometric interpolation problem, which similarly depend only on the total number of even and odd integers in the components of each of the vectors \( m \) and \( m' \) of (1.2). To this end, we use throughout the notation of (cf. (1.3))

\[
(1.8) \quad \begin{cases} 
E := \text{number of even integers in the components of } m = (0, m_2, \ldots, m_p), \\
O := \text{number of odd integers in the components of } m = (0, m_2, \ldots, m_p), \\
E' := \text{number of even integers in the components of } m' = (m'_1, m'_2, \ldots, m'_q), \\
O' := \text{number of odd integers in the components of } m' = (m'_1, m'_2, \ldots, m'_q),
\end{cases}
\]
so that

\[(1.9) \quad E + O = p \quad \text{and} \quad E' + O' = q.\]

2. Statements of Theorems 1 and 2.

With the notation of section 1, we now state our basic results, Theorem 1 (when \( N \) is odd) and Theorem 2 (when \( N \) is even). We begin with the case when \( N \) is odd.

**THEOREM 1.** Let \( N \) be odd, so that (cf. (1.6)) \( n =: 2r + 1 \) and \( p + q =: 2s + 1 \) are both odd. Then, a necessary condition for the \((m; m')\) 2-periodic trigonometric interpolation problem (1.5) to be regular is that (cf. (1.8))

\[(2.1) \quad E + E' - 1 = s = O + O'.\]

Moreover, if the components of \( m \) and \( m' \) are, respectively, alternately even and odd integers, i.e. (cf. (1.2) and (1.3)),

\[(2.2) m_i + m_{i+1} \quad \text{is odd} \quad (i = 0, 1, \ldots, p - 1), \quad \text{and} \quad m'_i + m'_{i+1} \quad \text{is odd} \quad (i = 0, 1, \ldots, q - 1),\]

then (2.1) is both necessary and sufficient for this \((m; m')\) 2-periodic trigonometric interpolation problem to be regular.

It is not yet known if condition (2.1) is, by itself, both necessary and sufficient for the regularity of this trigonometric interpolation problem when \( N \) is odd, though we tend to believe that it is. Indeed, with \( N \) odd and with (2.1) holding, there are examples showing that the trigonometric interpolation problem is regular, without (2.2) holding. Explicit examples of this will be given in section 4.

We continue with the case when \( N \) of (1.6) is even. Here, there are three cases to be dealt with:

\[(2.3) \quad \begin{cases} 
  i) & n \text{ is even } (n =: 2r), \text{ and } p + q \text{ is odd } (p + q =: 2s + 1), \\
  ii) & n \text{ is even } (n =: 2r), \text{ and } p + q \text{ is even } (p + q =: 2s + 2), \\
  iii) & n \text{ is odd } (n =: 2r + 1), \text{ and } p + q \text{ is even } (p + q =: 2s + 2). 
\]
Because $N$ is now even, the sought interpolant $t_M(x)$ is necessarily of the form (1.4'), and $\varepsilon(= 0$ or 1) must also be appropriately chosen in each case of (2.3). We also use the notation $[[x]]$ to denote the integer part of the real number $x$.

**Theorem 2.** i) Let $N$ be even, with (cf. (2.3i)) $n =: 2r$ even and $p + q =: 2s + 1$ odd. Then, a necessary condition for the $(m; m')$ 2-periodic trigonometric interpolation problem (1.5) to be regular is that

\[(2.4i) \quad E + E' - 1 = s = O + O'.\]

Moreover, if the alternation condition (2.2) holds, then (2.4i) is both necessary and sufficient for this $(m; m')$ 2-periodic trigonometric interpolation problem to be regular, with $\varepsilon$ of (1.4') being given by

\[(2.5i) \quad \varepsilon = \begin{cases} 
0, & \text{if } p \text{ is odd;} \\
1, & \text{if } p \text{ is even.} 
\end{cases}\]

ii) Let $N$ be even, with (cf. (2.3ii)) $n =: 2r$ even and $p + q =: 2s + 2$ even. Then, a necessary condition for the $(m; m')$ 2-periodic trigonometric interpolation problem (1.5) to be regular is that

\[(2.4ii) \quad E = [[(p+1)/2]], \text{ and } E' = [[(q+1)/2]].\]

Moreover, if the alternation condition (2.2) holds, then (2.4ii) is both necessary and sufficient for this $(m; m')$ 2-periodic trigonometric interpolation problem to be regular, with $\varepsilon$ of (1.4') being given by

\[(2.5ii) \quad \varepsilon = \begin{cases} 
0, & \text{if } p \text{ and } q \text{ are both odd;} \\
1, & \text{if } p \text{ and } q \text{ are both even.} 
\end{cases}\]

iii) Let $N$ be even, with (cf. (2.3iii)) $n =: 2r + 1$ odd and $p + q =: 2s + 2$ even. Then, a necessary condition for the $(m; m')$ 2-periodic trigonometric interpolation problem (1.5) to be regular is that

\[(2.4iii) \quad E + E' - (O + O') = 0 \text{ or } 2.\]
Moreover, if the alternation condition (2.2) holds, then (2.4iii) is both necessary and sufficient for this \((m; m')\) interpolation problem to be regular, with \(\varepsilon\) of (1.4') being given by

\[
(2.5iii) \quad \varepsilon = \begin{cases} 
0, & \text{if } E + E' = s + 2 \quad \text{and } \quad O + O' = s; \\
1, & \text{if } E + E' = s + 1 = O + O'.
\end{cases}
\]

Having stated above our main results, we next outline the remaining portions of this paper. In section 3, we give some useful results on determinants having special forms. Then in section 4, we derive in detail necessary and sufficient determinantal conditions for the regularity of this \((m; m')\) 2-periodic trigonometric interpolation problem when \(N\) is odd, from which the necessary condition of (2.1) of Theorem 1 is obtained. The remainder of section 4 gives a detailed treatment of the sufficiency part of Theorem 1 when \(N\) is odd. Because the methods used in establishing the results of Theorem 2 (when \(N\) is even), are similar to those used in establishing Theorem 1 (when \(N\) is odd), only the first part i) of Theorem 2 is given in detail in section 5.

To conclude this section, we now comment on the relationship of Theorem 1 and 2 here to known results in the literature. In [1], Cavaretta, Sharma, and Varga considered what can be analogously called the 1-periodic trigonometric interpolation problem on the equidistant nodes \(\{x_k(n)\}_{k=0}^{n-1}\), with \(x_k(n) := 2k\pi/n\), where the single vector

\[
(2.6) \quad m := (m_1, m_2, \cdots, m_q)
\]

has nonnegative integer components with \(0 =: m_1 < m_2 < \cdots < m_q\). Here, \(n\) is either an odd or even positive integer, and for arbitrary complex numbers \(\{\alpha_{k\nu}\}_{k=0}^{n-1}, \nu = 0, 1, \cdots, q\), the associated 1-periodic lacunary interpolation problem was defined by

\[
(2.7) \quad t_m^{m_{k\nu}}(x_k) = \alpha_{k\nu} \quad (k = 0, 1, \cdots, n - 1; \nu = 0, 1, \cdots, q),
\]

where \(t_M(x)\) is a trigonometric polynomial of the appropriate form (1.4) or (1.4'). In [1], necessary and sufficient conditions, for the interpolation problem (2.7) to be regular, were derived, and remarkably, these conditions depend only on the number of odd and even
integers in \( m \). (For a recent survey of lacunary trigonometric interpolation, see Chapter 11 of Lorentz, Jetter, and Riemenschneider [4].)

Curiously, if \( m = m' \) in (1.2), one would expect that this would cover completely the case treated in [1]. This, however, is not the case because the number of nodes in (1.1) is always even, namely \( 2n \), whereas the case treated in [1] allows the total number of nodes to be either odd or even. In this sense, the results in this paper can be viewed as being complementary to those of [1]. In addition, the necessary and sufficient conditions of Theorems 1 and 2 of this paper require the alternation condition of (2.2), a condition which doesn’t appear in [1].

3. Some Lemmas on Determinants.

In subsequent sections, we shall need results for deciding whether certain determinants are nonzero. For this purpose, we collect in this section some elementary but useful results on determinants. We begin with the following well known result for generalized Vandermonde determinants.

**Lemma 1.** Let \( \{t_j\}_{j=1}^s \) be positive real numbers with \( 0 < t_1 < t_2 < \cdots < t_s \), and let \( \{\mu_j\}_{j=1}^s \) be nonnegative integers with \( 0 \leq \mu_1 < \mu_2 < \cdots < \mu_s \). Then, the determinant \( D \) of order \( s \), defined by

\[
D := \begin{vmatrix}
 t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_s^{\mu_1} \\
 t_1^{\mu_2} & t_2^{\mu_2} & \cdots & t_s^{\mu_2} \\
 \vdots & \vdots & & \vdots \\
 t_1^{\mu_s} & t_2^{\mu_s} & \cdots & t_s^{\mu_s}
\end{vmatrix},
\]

(3.1)

satisfies \( D > 0 \). Similarly, if the positive real numbers \( \{t_j\}_{j=1}^s \) in (3.1) are strictly decreasing, then \( D \neq 0 \) with

\[
\text{sgn} D = (-1)^{s(s-1)/2}.
\]

(3.2)

**Proof.** The first part of Lemma 1 is of course the well known result for generalized Vandermonde determinants (cf. Gantmacher [2, p. 99]). For the second part of Lemma 1, assume \( \{t_j\}_{j=1}^s \) is a strictly decreasing sequence of positive real numbers. On interchanging the
columns of $D$ of (3.1) so as to achieve a rearrangement of $\{t_j\}_{j=1}^s$ which is a strictly increasing sequence of positive real numbers, then part 1 of Lemma gives that this new determinant is nonzero, and, on counting the number of column interchanges necessary to achieve this rearrangement, it follows that (3.2) is valid.

The next result is similar to the result of Lemma 2 of Cavaretta, Sharma, and Varga [1].

**Lemma 2.** Let $\{t_j\}_{j=1}^s$ be positive real numbers with $0 < t_1 < t_2 < \cdots < t_s$, and let $\{m_j\}_{j=1}^p$ and $\{m'_j\}_{j=1}^q$ be any two sets of nonnegative integers with $0 \leq m_1 < m_2 < \cdots < m_p$ and $0 \leq m'_1 < m'_2 < \cdots < m'_q$, where $p + q = s > 0$. Then, the determinant $D$ of order $s$, defined by

$$
D := \begin{vmatrix}
  t_1^{m_1} & t_2^{m_1} & \cdots & t_s^{m_1} \\
  \vdots & \vdots & & \vdots \\
  t_1^{m_p} & t_2^{m_p} & \cdots & t_s^{m_p} \\
  t_1^{m'_1} & -t_2^{m'_1} & \cdots & (\cdots) (-1)^{s-1} t_s^{m'_1} \\
  \vdots & \vdots & & \vdots \\
  t_1^{m'_q} & -t_2^{m'_q} & \cdots & (\cdots) (-1)^{s-1} t_s^{m'_q}
\end{vmatrix},
$$

(3.3)

satisfies $D \neq 0$ with

$$
\text{sgn } D = (-1)^{(q(s+p-1))/2}.
$$

**Proof.** The Laplace expansion of $D$ of (3.3) about its first $p$ rows, is given (cf. Karlin [3, p. 6]) by

$$
\sum_{1 \leq j_1 < \cdots < j_p \leq s} (-1)^{1+2+\cdots+p}+(j_1+j_2+\cdots+j_p) D_1 \left( \begin{array}{cccc}
  1 & 2 & \cdots & p \\
  j_1 & j_2 & \cdots & j_p
\end{array} \right) \cdot D_2 \left( \begin{array}{cccc}
  p+1 & p+2 & \cdots & p+q \\
  j'_1 & j'_2 & \cdots & j'_q
\end{array} \right),
$$

(3.4)

where $D_1 \left( \begin{array}{cccc}
  1 & 2 & \cdots & p \\
  j_1 & j_2 & \cdots & j_p
\end{array} \right)$ is the determinant formed from the rows $1, 2, \cdots, p$ and columns $j_1, j_2, \cdots, j_p$ of $D$ (with $1 \leq j_1 < j_2 < \cdots < j_p \leq s$), and $D_2 \left( \begin{array}{cccc}
  p+1 & p+2 & \cdots & p+q \\
  j'_1 & j'_2 & \cdots & j'_q
\end{array} \right)$
is the determinant formed from the complementary rows \(p + 1, p + 2, \ldots, p + q\) and complementary columns \(j_1', j_2', \ldots, j_q'\) of \(D\) (with \(1 \leq j_1' < j_2' < \cdots < j_q' \leq s\)). From part 1 of Lemma 1, \(\text{sgn } D_1 \left( \begin{array}{cccc} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{array} \right) = 1\), and, on factoring out \((-1)^{j_k-1}\) from each \(k\)-th column of \(D_2 \left( \begin{array}{cccc} p + 1, & p + 2, & \cdots, & p + q \\ j_1', & j_2', & \cdots, & j_q' \end{array} \right)\), part 1 of Lemma 1 similarly gives that

\[
\text{sgn } D_2 \left( \begin{array}{cccc} p + 1, & p + 2, & \cdots, & p + q \\ j_1', & j_2', & \cdots, & j_q' \end{array} \right) = (-1)^{j_1' + j_2' + \cdots + j_q' - q}.
\]

It then follows that each term of the sum (3.5) is nonzero with the same sign, this sign being given in (3.4). \(\square\)

We remark that Lemma 2 is also valid when the set \(\{m_j^j\}_{j=1}^q\) is empty (i.e., \(q = 0\)) since Lemma 2 reduces to the first part of Lemma 1 in this case. Similarly, Lemma 2 is also valid when the set \(\{m_j\}_{j=1}^p\) is empty (i.e., \(p = 0\)).

**LEMMA 3.** Let \(\{t_j\}_{j=1}^k\) be positive real numbers with \(t_1 > t_2 > \cdots > t_k > 0\), let \(\{u_j\}_{j=1}^\ell\) be positive real numbers with \(0 < u_1 < u_2 < \cdots < u_\ell\), and let \(k + \ell =: s > 0\). If \(\{\mu_j\}_{j=0}^s\) are nonnegative integers with \(0 \leq \mu_1 < \mu_2 < \cdots < \mu_s\), then the determinant \(D_s\) of order \(s\), defined by

\[
D_s := \begin{vmatrix}
 t_1^{\mu_1} & t_1^{\mu_2} & \cdots & t_1^{\mu_s} & u_1^{\mu_1} & u_1^{\mu_2} & \cdots & u_1^{\mu_s} \\
 t_2^{\mu_1} & t_2^{\mu_2} & \cdots & t_2^{\mu_s} & -u_1^{\mu_2} & -u_2^{\mu_2} & \cdots & -u_\ell^{\mu_2} \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 t_k^{\mu_1} & t_k^{\mu_2} & \cdots & t_k^{\mu_s} & \sigma u_1^{\mu_s} & \sigma u_2^{\mu_s} & \cdots & \sigma u_\ell^{\mu_s}
\end{vmatrix}
\]

(\(\sigma := (-1)^{s-1}\)),

satisfies \(D_s \neq 0\) with

\[
\text{sgn } D_s = (-1)^{s(s-1)/2}.
\]

(3.7)
Under the same hypotheses, if
\[
\tilde{D}_s := \begin{vmatrix}
  t_{i_1}^{\mu_1} & t_{i_2}^{\mu_2} & \cdots & t_{i_k}^{\mu_k} & u_1^{\mu_1} & u_2^{\mu_2} & \cdots & u_\xi^{\mu_\xi} \\
  -t_{i_1}^{\nu_1} & -t_{i_2}^{\nu_2} & \cdots & -t_{i_k}^{\nu_k} & u_1^{\nu_1} & u_2^{\nu_2} & \cdots & u_\xi^{\nu_\xi} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  \sigma t_{i_1}^{\nu_1} & \sigma t_{i_2}^{\nu_2} & \cdots & \sigma t_{i_k}^{\nu_k} & u_1^{\nu_1} & u_2^{\nu_2} & \cdots & u_\xi^{\nu_\xi} \\
\end{vmatrix}
\quad (\sigma := (-1)^{s-1}),
\]
then
\[
\tilde{D}_s > 0.
\]

**Proof.** Here, we use the Laplace expansion of \( D_s \) about its first \( k \) columns. Then as in (3.5),
\[
(3.10) \quad D_s = \sum_{1 \leq i_1 < \cdots < i_k \leq s} \left( \frac{1+2+\cdots+k}{i_1, i_2, \ldots, i_k} \right) D_1 \left( \begin{array}{c}
  i_1, i_2, \ldots, i_k \\
  1, 2, \ldots, k
\end{array} \right) D_2 \left( \begin{array}{c}
  i_1', i_2', \ldots, i_\ell' \\
  k+1, k+2, \ldots, k+\ell
\end{array} \right).
\]

From the second part of Lemma 1, it follows from (3.2) and the hypotheses on \( \{ t_j \}_{j=1}^k \) that
\[
\text{sgn } D_1 \left( \begin{array}{c}
  i_1, i_2, \ldots, i_k \\
  1, 2, \ldots, k
\end{array} \right) = (-1)^{\frac{k(k-1)}{2}},
\]
while, on factoring out \((-1)^{\nu-1}\) from each \( \nu \)-th row of \( D_2 \), it similarly follows from the first part of Lemma 1 that
\[
\text{sgn } D_2 \left( \begin{array}{c}
  i_1', i_2', \ldots, i_\ell' \\
  k+1, k+2, \ldots, k+\ell
\end{array} \right) = (-1)^{\sum_{\nu=1}^{\ell} (\nu-1)}.
\]

Thus, each term of the sum (3.10) is nonzero with the same sign, this sign being given in (3.7). Finally, for the determinant \( \tilde{D}_s \) of (3.8), simply factoring out \((-1)^{j-1}\) from each \( j \)-th row \((j = 1, 2, \ldots, s)\), exactly gives the determinant \( D_s \) of (3.6). Thus, we have \( \tilde{D}_s = (-1)^{s(s-1)/2} D_s \), and with (3.7), it follows that \( \tilde{D}_s > 0 \). \( \square \)

As in the remark following Lemma 2, we note that both parts of Lemma 3 are valid when either the set \( \{ u_\ell \}_{\ell=1}^s \) is empty (i.e., \( \ell = 0 \)) or when the set \( \{ t_j \}_{j=1}^k \) is empty (i.e., \( k = 0 \)).

With the definition of (1.6), we assume here that $N = n(p + q)$ is odd, so that $n$ and $p + q$ are also both odd, and we write

$$n = 2r + 1 \text{ and } p + q = 2s + 1.$$ 

In this case, we have (cf. (1.7))

$$M = \frac{N - 1}{2} = ns + r,$$

and the sought trigonometric polynomials must be of the form (1.4).

The derivation of necessary and sufficient conditions for a unique trigonometric interpolant in (1.5) is based on the elementary notion that the interpolation of null data is unique if and only if certain associated determinants are nonzero. To begin, we use the familiar device of identifying (in a 1-1 way) trigonometric polynomials (in the real variable $x$) with algebraic polynomials (in the complex variable $z$), through the transformation $z = e^{ix}$. Then, any trigonometric polynomial of the form (1.4) can be expressed as

$$t_M(x) = z^{-M}q_{2M}(z) \quad (q_{2M} \in \pi_{2M}),$$

where $\pi_k$ denotes the set of all complex algebraic polynomials of degree at most $k$. Thus, with (4.1) and (4.2), we can express (4.3) as

$$t_M(x) = z^{-M} \sum_{\lambda=0}^{2s} z^{\lambda n} Q_\lambda(z),$$

where

$$Q_\lambda(z) := \sum_{j=0}^{n-1} a_{\lambda,j} z^j \quad (\lambda = 0, 1, \cdots, 2s).$$

Then, since $z = e^{ix}$ implies $d/dx = izd/dz$, the interpolation conditions of (1.5), applied to
(4.4) for null data, reduce, respectively with $$z_{2k} := e^{i\pi 2k}$$ and $$z_{2k+1} := e^{i\pi 2k+1}$$, to

$$\begin{align*}
\sum_{j=0}^{n-1} z_{2k}^j \left[ \sum_{\lambda=0}^{2\nu} a_{\lambda,j} (j + \lambda n - M)^{m_{\nu}} \right] &= 0, \quad (\nu = 1, 2, \ldots, p; \ k = 0, 1, \ldots, n - 1), \\
\sum_{j=0}^{n-1} z_{2k+1}^j \left[ \sum_{\lambda=0}^{2\nu} (-1)^{\lambda} a_{\lambda,j} (j + \lambda n - M)^{m_{\nu}} \right] &= 0, \quad (\nu = 1, 2, \ldots, q; \ k = 0, 1, \ldots, n - 1),
\end{align*}$$

(4.5)

where we have used the facts (cf. (1.1)) that $$z_{2k}^n = 1$$ and $$z_{2k+1}^n = -1$$. Next, if we define the quantity in square brackets in the first display of (4.5) to be $$B_j$$ and if we set $$p_{n-1}(z) := \sum_{j=0}^{n-1} B_j z^j$$, then (4.5) implies that $$p_{n-1}(z_{2k}) = 0$$ for $$k = 0, 1, \ldots, n - 1$$. But as the $$n$$ associated nodes $$\{z_{2k}\}_{k=1}^{n}$$ are all distinct, then $$p_{n-1}(z) \equiv 0$$; whence, $$B_j = 0$$ for all $$j = 0, 1, \ldots, n - 1$$.

As the same argument applies to the second display of (4.5), we thus have

$$\begin{align*}
\sum_{\lambda=0}^{2\nu} a_{\lambda,j} (j + \lambda n - M)^{m_{\nu}} &= 0, \quad (\nu = 1, 2, \ldots, p; \ j = 0, 1, \ldots, n - 1), \\
\sum_{\lambda=0}^{2\nu} (-1)^{\lambda} a_{\lambda,j} (j + \lambda n - M)^{m_{\nu}} &= 0, \quad (\nu = 1, 2, \ldots, q; \ j = 0, 1, \ldots, n - 1).
\end{align*}$$

(4.6)

Recalling from (4.2) that $$M = ns + r$$ and from (4.1) that $$n = 2r + 1$$, then on dividing out a factor of $$n^{m_{\nu}}$$ and $$n^{m_{\nu}}$$, respectively, from each of the two equations of (4.6), and on setting

$$\alpha_j := \frac{j - r}{2r + 1} \quad (j = 0, 1, \ldots, 2r),$$

(4.7)

it follows from (4.6) that, for each $$j = 0, 1, \ldots, 2r$$,

$$\begin{align*}
\sum_{\lambda=0}^{2\nu} a_{\lambda,j} (\alpha_j + \lambda - s)^{m_{\nu}} &= 0 \quad (\nu = 1, 2, \ldots, p), \\
\sum_{\lambda=0}^{2\nu} (-1)^{\lambda} a_{\lambda,j} (\alpha_j + \lambda - s)^{m_{\nu}} &= 0 \quad (\nu = 1, 2, \ldots, q).
\end{align*}$$

(4.8)

But for each choice of $$j = 0, 1, \ldots, 2r$$, the above equations represent $$2s + 1$$ homogeneous linear equations in the $$2s + 1$$ variables $$\{a_{\lambda,j}\}_{\lambda=0}^{2\nu}$$. If $$\Delta(\alpha_j)$$ denotes the determinant of order $$2s + 1$$ of the coefficients of $$\{a_{\lambda,j}\}_{\lambda=0}^{2\nu}$$ in (4.8), then $$\Delta(\alpha_j) \neq 0$$ for each $$j = 0, 1, \ldots, n - 1$$ if and only if $$t_M(x) \equiv 0$$ in (4.3). Consequently, we immediately have the result of
**Proposition 1.** Let \( N = n(p + q) \) be odd, so that \( n \) is odd (with \( n = 2r + 1 \)) and \( p + q \) is odd (with \( p + q = 2s + 1 \)). Then, in order for the \((m; m')\) 2-periodic trigonometric interpolation problem (1.5) to be regular on the \( 2n \) equidistant nodes \( \{x_k\}_{k=0}^{2n-1} \) in \([0, 2\pi)\), it is necessary and sufficient that

\[
\Delta(\alpha_j) \neq 0 \quad (\alpha_j := \frac{j - r}{2r + 1}; \ j = 0, 1, \ldots, 2r),
\]

where \( \Delta(\alpha_j) \) denotes the determinant of order \( 2s + 1 \) of the coefficients of \( \{a_{\lambda,j}\}_{\lambda=0}^{2s} \) in (4.8).

We remark from (4.7) that the \( \alpha_j \)'s of (4.7) all satisfy

\[
|\alpha_j| < \frac{1}{2} \quad (j = 0, 1, \ldots, 2r).
\]

Next, for convenience, let \( \Delta(\alpha) \) denote the determinant (of order \( 2s + 1 \)) of the coefficients of \( \{a_{\lambda,j}\}_{\lambda=0}^{2s} \) in (4.8) when \( \alpha_j \) is replaced by the variable \( \alpha \). Then, this determinant \( \Delta(\alpha) \) can be expressed in usual row-column notation as

\[
\Delta(\alpha) = |c_{i,j}(\alpha)| \quad (1 \leq i, j \leq 2s + 1),
\]

where, from (4.8),

\[
c_{i,j}(\alpha) := \begin{cases} 
(\alpha + j - 1 - s)^{m_i} & (1 \leq i \leq p; 1 \leq j \leq 2s + 1), \\
(\alpha + j - 1 - s)^{m_i-p} & (p + 1 \leq i \leq p + q = 2s + 1; 1 \leq j \leq 2s + 1).
\end{cases}
\]

(4.12)

Thus, the determinant \( \Delta(\alpha) \) is explicitly given by

\[
\Delta(\alpha) = \begin{vmatrix} 
1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
(\alpha - s)^{m_2} & (\alpha + 1 - s)^{m_2} & \alpha^{m_2} & (\alpha + 1)^{m_2} & (\alpha + s)^{m_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(\alpha - s)^{m_2} & (\alpha + 1 - s)^{m_2} & \alpha^{m_2} & (\alpha + 1)^{m_2} & (\alpha + s)^{m_2} \\
(\alpha - s)^{m_2} & (\alpha + 1 - s)^{m_2} & \alpha^{m_2} & (\alpha + 1)^{m_2} & (\alpha + s)^{m_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(\alpha - s)^{m_2} & (\alpha + 1 - s)^{m_2} & \alpha^{m_2} & (\alpha + 1)^{m_2} & (\alpha + s)^{m_2} \\
(\alpha - s)^{m_2} & (\alpha + 1 - s)^{m_2} & \alpha^{m_2} & (\alpha + 1)^{m_2} & (\alpha + s)^{m_2}
\end{vmatrix},
\]

(4.13)
where the first $p$ rows all have entries with positive signs, while the last $q$ rows have entries which are alternately positive and negative in sign.

Next, we derive a useful general relationship between $\Delta(\alpha)$ and $\Delta(-\alpha)$. Starting with $\Delta(-\alpha)$ from (4.13), interchange its first and last $(2s + 1)$st columns, its second and the $2s$th columns, etc., a total of $s$ interchanges. This gives from (4.13) that

$$
\begin{vmatrix}
1 & \cdots & 1 & 1 & \cdots & 1 \\
(-\alpha + s)^m & (-\alpha + 1)^m & (-\alpha)^m & (-\alpha - s)^m \\
\vdots & & & & & \\
\vdots & & & & & \\
(-\alpha + s)^m & (-\alpha + 1)^m & (-\alpha)^m & (-\alpha - s)^m \\
\end{vmatrix}
$$

$$
\Delta(-\alpha) = (-1)^s
\begin{vmatrix}
(-\alpha + s)^m & (-\alpha + 1)^m & (-\alpha)^m & (-\alpha - s)^m \\
(-\alpha + s)^m & (-\alpha + 1)^m & (-\alpha)^m & (-\alpha - s)^m \\
\vdots & & & & \\
\vdots & & & & \\
(-\alpha + s)^m & (-\alpha + 1)^m & (-\alpha)^m & (-\alpha - s)^m \\
\end{vmatrix}
$$

If $m_j$ is even, the $j$-th row of the determinant above is the same as the $j$-th row of $\Delta(\alpha)$ of (4.13), the same being true for the appropriate rows of these determinants if $m'_j$ is even. But if $m_j$ is odd, the $j$-th row of the determinant above is the negative of the $j$-th of $\Delta(\alpha)$ of (4.13), the analogous property holding when $m'_j$ is odd. On factoring out $(-1)$ from each row where $m_j$ or $m'_j$ is odd, it follows from the definition of (1.8) that

$$
(4.14) \quad \Delta(-\alpha) = (-1)^{s+0+0'} \Delta(\alpha) \quad (\alpha \text{ arbitrary}).
$$

It is evident from (4.7) that $\alpha_s = 0$. From Proposition 1, it thus follows that $\Delta(0) \neq 0$ is a necessary condition for this $(m; m')$ 2-periodic interpolation problem (1.5) to be regular in the case when $N$ is odd. Assuming that $m'_1 > 0$ (the treatment of the case $m'_1 = 0$ being similar), take the case $\alpha = 0$ of the determinant of (4.13) (of order $2s - 1$), and subtract its first column from its last $(2s + 1)$-st column, its second column from its $2s$-th column, etc.
This leaves $\Delta(0)$ unchanged and gives

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
-(s)^{m_2} & -(s+1)^{m_2} & 0 & 1-(s)^{m_2} & s^{m_2}-(s)^{m_2} \\
\vdots & \vdots & & \vdots & \vdots \\
-(s)^{m_s} & -(s+1)^{m_s} & \cdots & 0 & 1-(s)^{m_s} & s^{m_s}-(s)^{m_s} \\
-(s)^{m'_1} & -(s+1)^{m'_1} & \cdots & 0 & (s-1)^{s-1}[1-(s)^{m'_1}] & s^{m'_1}-(s)^{m'_1} \\
\vdots & \vdots & & \vdots & \vdots \\
-(s)^{m'_s} & -(s+1)^{m'_s} & \cdots & 0 & (s-1)^{s-1}[1-(s)^{m'_s}] & s^{m'_s}-(s)^{m'_s}
\end{vmatrix}
\]

(4.15) \ 
\Delta(0) =

Note that the last $s$ columns of $\Delta(0)$ in (4.15) have a zero row whenever $m_j$ or $m'_j$ is even. Thus, the last $s$ columns of $\Delta(0)$ have, with the notation of (1.8), $E + E'$ zero rows, while the last $s+1$ columns of $\Delta(0)$ have $E + E' - 1$ zero rows. Next, assume that $E + E' \geq s + 2$, so that the last $s+1$ columns of $\Delta(0)$ have at least $s+1$ zero rows. Now, the Laplace expansion of $\Delta(0)$, in terms of its first $s$ columns, is given by the sum of (3.10) with $k := s$ and $\ell := s+1$. But, the fact that the last $s+1$ columns of $\Delta(0)$ have at least $s+1$ zero rows implies that the final determinant $D_2 \left( \begin{smallmatrix} i_1' & i_2' & \cdots & i_{s+1}' \\ s+1 & s+2 & \cdots & 2s+1 \end{smallmatrix} \right)$ of (3.10) must vanish, since it has at least one zero row, for any choice of $1 \leq i_1' < i_2' < \cdots < i_{s+1}' \leq 2s+1$. Hence, $E + E' \geq s + 2$ implies $\Delta(0) = 0$. Thus, a necessary condition for regularity is that

\[E + E' \leq s + 1.\]

In a similar fashion (on adding appropriate columns of $\Delta(0)$ in (4.15)), we deduce that a necessary condition for regularity is that (cf. (1.8))

\[O + O' \leq s.\]

But since $(E + E') + (O + O') = p + q = 2s + 1$ from (1.9) and (4.1), it must follow from the above two inequalities that

(4.16) \ 
\[E + E' = s + 1 \text{ and } O + O' = s,\]

which establishes the necessary condition (2.1) of Theorem 1.
We remark that (4.16) is a necessary condition for the regularity of this \((m; m')\) 2-periodic interpolation problem when \(N\) is odd. In fact, we now show that (4.16) implies \(\Delta(0) \neq 0\) when \(N\) is odd. Because the derivatives in the interpolation conditions of (1.5) can be applied in any order, we may assume (without loss of generality) that

\[
\begin{align*}
0 = m_1, m_2, \cdots, m_\ell \ & \text{are even, and} \ m_{\ell+1}, m_{\ell+2}, \cdots, m_p \ & \text{are odd, and} \\
m_1', m_2', \cdots, m_k' \ & \text{are even, and} \ m_{k+1}', m_{k+2}', \cdots, m_q' \ & \text{are odd,}
\end{align*}
\]

(4.17)

where, from (4.16),

\[
\ell + k - 1 = s = p - \ell + q - k.
\]

(4.18)

From our previous construction, we know that the last \(s\) columns of \(\Delta(0)\) of (4.15) have \(s\) rows of zeros, corresponding to the even integers \(m_2, \cdots, m_\ell, m_1', \cdots, m_k'\). Thus, the Laplace expansion of \(\Delta(0)\) of (4.15) about its center column, namely \((1, 0, \cdots, 0)^T\), results in a single determinant, of order \(2s\), equal to \(\pm \Delta(0)\), which can be expressed (up to a nonzero constant factor) as the product

\[
\Delta(0) = D_1 \cdot D_2,
\]

(4.19)

where, after elementary column operations, \(D_1\) and \(D_2\) are determinants, each of order \(s\), given by

\[
D_1 = \begin{vmatrix}
1 & 2^{m_2} & \cdots & (s-1)^{m_2} & s^{m_2} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
1 & 2^{m_\ell} & \cdots & (s-1)^{m_\ell} & s^{m_\ell} \\
(-1)^{s-1} & (-1)^s 2^{m_1'} & \cdots & -(s-1)^{m_1'} & s^{m_1'} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
(-1)^{s-1} & (-1)^s 2^{m_k'} & \cdots & -(s-1)^{m_k'} & s^{m_k'}
\end{vmatrix}
\]

(4.20)
and

\[
D_2 = \begin{pmatrix}
1 & 2^{m_{t+1}} & \cdots & (s-1)^{m_{t+1}} & s^{m_{t+1}} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & 2^{m_p} & \cdots & (s-1)^{m_p} & s^{m_p} \\
(-1)^{s-1} & (-1)^s 2^{m_{k+1}} & \cdots & -(s-1)^{m_{k+1}} & s^{m_{k+1}} \\
\vdots & \vdots & & \vdots & \vdots \\
(-1)^{s-1} & (-1)^s 2^{m_q} & \cdots & -(s-1)^{m_q} & s^{m_q}
\end{pmatrix}
\]

Upon multiplying all entries in the last \( k \) rows of \( D_1 \) and the last \( q - k \) rows of \( D_2 \) by \((-1)^{s-1}\), then Lemma 2 directly shows that both \( D_1 \) and \( D_2 \) are nonzero. Thus (cf. (4.19)), \( \Delta(0) \neq 0 \). This gives us the following result, which establishes the first part of Theorem 1.

**Proposition 2.** Let \( N = n(p+q) \) be odd, so that \( n \) is odd (with \( n = 2r + 1 \)) and \( p + q \) is odd (with \( p + q = 2s + 1 \)). Then, a necessary and sufficient condition that \( \Delta(0) \neq 0 \) is that (cf. (1.8))

\[
E + E' - 1 = s = O + O'.
\]

(4.22)

We now complete the proof of Theorem 1. We assume that (4.22) is valid, and that the components of \( m \) and \( m' \) are, respectively, alternately even and odd integers (cf. (2.2)). To establish that this \((m,m')\)-periodic trigonometric interpolation problem is regular, it is sufficient, from Proposition 1 and (4.10), to show that \( \Delta(\alpha) \neq 0 \) for any real \( \alpha \) with \( 0 < |\alpha| < 1/2 \). But from (4.22), we have \( O + O' = s \), which implies from (4.14) that \( \Delta(-\alpha) = \Delta(\alpha) \). Thus, to establish Theorem 1, it suffices to show that

\[
\Delta(\alpha) \neq 0 \text{ for any } 0 < \alpha < 1/2.
\]

(4.23)

Continuing, since \( p + q = 2s + 1 \) is odd from (4.1) when \( N \) is odd, assume that \( p \) is even, i.e., \( p = 2\ell \), and that \( q \) is odd, i.e., \( q = 2k + 1 \) (the remaining case of \( p \) odd and \( q \) even being similar). With the alternation hypothesis of (2.2), then

\[
\begin{align*}
0 = m_1 < m_3 < \cdots < m_{2\ell-1} & \text{ are even integers, and} \\
m_2 < m_4 < \cdots < m_{2\ell} & \text{ are odd integers.}
\end{align*}
\]

(4.24)
Next, with the alternation hypothesis of (2.2), assume that

\[(4.24') \quad \begin{cases} m_1' < m_3' < \cdots < m_{2k+1}' & \text{are even integers, and} \\ m_2' < m_4' < \cdots < m_{2k}' & \text{are odd integers;} \end{cases} \]

(the remaining case of \(m_1'\) odd again being similar). From (4.24) and (4.24'), we have (cf. (1.8)) that \(E = \ell, \quad O = \ell, \quad E' = k + 1,\) and \(O' = k,\) so that (2.1) is satisfied.

With the hypotheses of (4.24) and (4.24'), the determinant \(\Delta(\alpha)\) of (4.13), of order \(2s + 1,\) can now be brought into a more suitable form. Indeed, if we factor out \((-1)^j\) from each \(j\)-th column \((j = 1, 2, \cdots, 2s + 1)\) of \(\Delta(\alpha)\) of (4.13), and if we factor out \((-1)^i\) from each \(i\)-th row of the first \(p\) rows \((i = 1, 2, \cdots, p)\) and change the signs in each of the remaining rows \((i = p + 1, p + 2, \cdots, p + q)\) of the resulting determinant, we obtain the determinant \(\hat{\Delta}(\alpha),\) of order \(2s + 1,\) given by

\[
\hat{\Delta}(\alpha) := \begin{vmatrix}
1 & -1 & \cdots & (-1)^s & (-1)^{s+1} & \cdots & 1 \\
(s - \alpha)^{m_2} & -(s - 1 - \alpha)^{m_2} & (-1)^{s+1}\alpha^{m_2} & (-1)^{s+2}(1 + \alpha)^{m_2} & -(s + \alpha)^{m_2} \\
(s - \alpha)^{m_3} & -(s - 1 - \alpha)^{m_3} & (-1)^{s}\alpha^{m_3} & (-1)^{s+1}(1 + \alpha)^{m_3} & (s + \alpha)^{m_3} \\
\vdots & & \vdots & & & \vdots & \\
(s - \alpha)^{m_p} & -(s - 1 - \alpha)^{m_p} & (-1)^{s+1}\alpha^{m_p} & (-1)^{s+2}(1 + \alpha)^{m_p} & -(s + \alpha)^{m_p} \\
(s - \alpha)^{m_1'} & (s - 1 - \alpha)^{m_1'} & \alpha^{m_1'} & (1 + \alpha)^{m_1'} & -(s + \alpha)^{m_1'} \\
-(s - \alpha)^{m_2'} & -(s - 1 - \alpha)^{m_2'} & \alpha^{m_2'} & (1 + \alpha)^{m_2'} & (s + \alpha)^{m_2'} \\
\vdots & & \vdots & & \vdots & \\
(s - \alpha)^{m_q'} & (s - 1 - \alpha)^{m_q'} & \alpha^{m_q'} & (1 + \alpha)^{m_q'} & (s + \alpha)^{m_q'}
\end{vmatrix}
\]

\[(4.25)\]

where \(\hat{\Delta}(\alpha)\) differs from \(\Delta(\alpha)\) by a multiplicative factor of \(\pm 1.\) Thus, we must show that \(\hat{\Delta}(\alpha) \neq 0\) for any \(\alpha\) with \(0 < \alpha < 1/2.\) In fact, we shall show that

\[(4.26) \quad \text{sgn}\hat{\Delta}(\alpha) = 1 \text{ for any } \alpha \text{ with } 0 < \alpha < 1/2.\]
Applying the Laplace expansion of $\hat{\Delta}(\alpha)$ in terms of its first $p$ rows, then

$$\hat{\Delta}(\alpha) = \sum_{1 \leq j_1 < \cdots < j_p \leq 2s+1} (-1)^{\frac{s^{p+1}}{2}} \prod_{\nu=1}^p D\begin{pmatrix} 1, 2, \cdots, p \\ j_1, j_2, \cdots, j_p \end{pmatrix} D\begin{pmatrix} p+1, \cdots, p+q \\ j_1', \cdots, j_q' \end{pmatrix},$$

(4.27)

where $(j'_1, j'_2, \cdots, j'_q)$ is complementary to $(j_1, j_2, \cdots, j_p)$ with $1 \leq j'_1 < j'_2 < \cdots < j'_q \leq 2s+1$.

Now, the determinant $D\begin{pmatrix} p+1, \cdots, p+q \\ j'_1, \cdots, j'_q \end{pmatrix}$ can be seen, from (4.25), to be exactly of the form of the determinant in (3.8) of Lemma 3, where $k$ (possibly zero) in (3.8) is the number of indices of $(j'_1, \cdots, j'_q)$ less than $s+1$. (It is here where the remark after Lemma 3 applies.) Note also that the strict monotonicity requirements, for the positive real numbers $t_j$'s and $u_j$'s in (3.8), follow from (4.25) and the fact that $0 < \alpha < 1/2$. Thus, from (3.9) of Lemma 3,

$$D\begin{pmatrix} p+1, \cdots, p+q \\ j'_1', \cdots, j'_q' \end{pmatrix} > 0.$$

(4.28)

Next, consider $D\begin{pmatrix} 1, 2, \cdots, p \\ j_1, j_2, \cdots, j_p \end{pmatrix}$ of (4.27). On factoring out $(-1)^{k-1}$ from the $k$th column of this determinant (where $k = 1, 2, \ldots, p$), we have

$$D\begin{pmatrix} 1, 2, \cdots, p \\ j_1, j_2, \cdots, j_p \end{pmatrix} = (-1)^{\sum_{k=1}^p (j_k-1)} D^*\begin{pmatrix} 1, 2, \cdots, p \\ j_1, j_2, \cdots, j_p \end{pmatrix},$$

(4.29)

where the determinant $D^*\begin{pmatrix} 1, 2, \cdots, p \\ j_1, j_2, \cdots, j_p \end{pmatrix}$ is exactly of the form of the determinant of (3.6) of Lemma 3, where $k$ (possibly zero) in (3.6) is the number of indices of $(j_1, \cdots, j_p)$ less than $s+1$. Again, the monotonicity requirements for applying (3.7) of Lemma 3 are fulfilled, and so from (3.7),

$$\text{sgn} D^*\begin{pmatrix} 1, 2, \cdots, p \\ j_1, j_2, \cdots, j_p \end{pmatrix} = (-1)^{p(p-1)/2}.$$

(4.30)
Hence, with (4.28)-(4.30), it follows that each term of the sum in (4.27) has the same sign, given by
\[
\text{sgn} \hat{\Delta}(\alpha) = (-1)^{\frac{n(p+1)}{2} + 2 \sum_{\nu=1}^{p} j_{\nu} - p + \frac{p(p-1)}{2}} = 1 \quad (0 < \alpha < 1/2),
\]
which establishes (4.26) and Theorem 1. \(\Box\)

We conclude this section with examples showing that regularity, in this case when \(N\) is odd, can hold with (2.1) but without the alternation condition (2.2) holding. Specifically, assume again that \(N = n(p+q)\) is odd, so that \(n\) is odd and \(p+q := 2s+1\) is odd. Choose \(s = 1\), and set
\[
(4.31) \quad m := (0, 2\ell) \quad \text{and} \quad m' := (2k+1) \quad \text{(any } \ell \geq 1, \text{ any } k \geq 0). 
\]

Now, (2.1) is satisfied (with \(E + E' - 1 = 1 = O + O'\)), but the alternation condition of (2.2) clearly does not hold. We shall show that this \((m, m')\) 2-periodic trigonometric interpolation problem defined by (4.31) is however regular. In this case, \(\hat{\Delta}(\alpha)\) is given by (cf. (4.13))
\[
(4.32) \quad -\Delta(\alpha) = \begin{vmatrix}
1 & 1 & 1 \\
(1-\alpha)^{2\ell} & \alpha^{2\ell} & (1+\alpha)^{2\ell} \\
(1-\alpha)^{2k+1} & \alpha^{2k+1} & -(1+\alpha)^{2k+1}
\end{vmatrix},
\]
where, from (4.14), \(\Delta(-\alpha) = \Delta(\alpha)\). Moreover, it is readily seen, on expanding \(\Delta(\alpha)\), that
\[
(4.33) \quad -\Delta(\alpha) = [(1-\alpha)^{2\ell} - \alpha^{2\ell}] [(1+\alpha)^{2k+1} + \alpha^{2k+1}] + [(1-\alpha)^{2k+1} - \alpha^{2k+1}] [(1+\alpha)^{2\ell} - \alpha^{2\ell}].
\]
But for any \(\ell \geq 1, k \geq 0,\) and \(\alpha\) with \(0 \leq \alpha < 1/2,\) the quantities in brackets are all positive, and this establishes that these \((0, 2\ell); (2k+1))\) 2-periodic interpolation problems are regular.

5. Necessary and Sufficient Conditions for Regularity: \(N\) Even.

If (cf. (1.6)) \(N = n(p+q)\) is even, then one of the three conditions of (2.3) necessarily holds, and we first consider the case (cf. (2.3i)) when
\[
(5.1) \quad n \text{ is even } (n = 2r), \text{ and } p+q \text{ is odd } (p+q = 2s+1),
\]
so that (cf. (1.7))

\[ M = N/2 = ns + r. \]  

Since \( N \) is even, we necessarily consider trigonometric interpolants of the form (1.4'), where \( \varepsilon (= 0 \text{ or } 1) \) in (1.4') is to be appropriately chosen. Following the procedure of §4 of using the transformation \( z = e^{i\varepsilon} \), we can similarly write any \( t_M(x) \) of the form (1.4') as

\[ t_M(x) = z^{-M} \left\{ \sum_{\lambda=0}^{2s} z^{n\lambda} Q_{\lambda}(z) + cz^{(2s+1)n} \right\}, \]  

where

\[ Q_{\lambda}(z) := \sum_{j=0}^{n-1} a_{\lambda,j} z^j \quad (\lambda = 0, 1, \ldots, 2s), \]

and where, because of (1.4'), we have

\[ a_{0,0} + c(-1)^{r+1} = 0. \]

With

\[ \alpha_j := \frac{j-r}{2r} \quad (j = 0, 1, \ldots, 2r), \]

we obtain, exactly as in (4.8) in the interpolation of null data, the following homogeneous 2s + 1 linear equations

\[ \begin{cases} \sum_{\lambda=0}^{2s} a_{\lambda,j} (\alpha_j + \lambda - s)^{m_\nu} = 0 \quad (\nu = 1, 2, \ldots, p), \\ \sum_{\lambda=0}^{2s} (-1)^{\lambda} a_{\lambda,j} (\alpha_j + \lambda - s)^{m_\nu} = 0 \quad (\nu = 1, 2, \ldots, q), \end{cases} \]  

in the 2s + 1 variables \( \{a_{\lambda,j}\}_{\lambda=0}^{2s} \), for each \( j = 1, 2 \cdots, n - 1 \). The departure from the case treated in §4 comes in the case \( j = 0 \) which involves the unknown constant \( c \) of (5.3). For this case \( j = 0 \), the associated homogeneous 2s + 2 linear equations for the unknowns \( c \) and \( \{a_{\lambda,0}\}_{\lambda=0}^{2s} \) are given by (5.4) and

\[ \begin{cases} \sum_{\lambda=0}^{2s} a_{\lambda,0} (\alpha_0 + \lambda - s)^{m_\nu} + c(\alpha_0 + s + 1)^{m_\nu} = 0 \quad (\nu = 1, 2, \ldots, p), \\ \sum_{\lambda=0}^{2s} (-1)^{\lambda} a_{\lambda,0} (\alpha_0 + \lambda - s)^{m_\nu} - c(\alpha_0 + s + 1)^{m_\nu} = 0 \quad (\nu = 1, 2, \ldots, q). \end{cases} \]
Clearly, the system of equations (5.6) is the same as those in (4.8), except that here we have in (5.5) that \( \alpha_j = \frac{i - r}{2r} \), whereas in (4.8), \( \alpha_j = \frac{j - r}{2r + 1} \). If, as in §4, \( \Delta(\alpha_j) \) denotes the determinant, of order \( 2s + 1 \), of the homogeneous system (5.6) for \( j = 1, 2, \ldots, n - 1 \), and if \( \Delta_0 \) denotes the determinant, of order \( 2s + 2 \), of the homogeneous system (5.7) and (5.4) for \( j = 0 \), then in complete analogy with Proposition 1 of §4, we have

**Proposition 3.** Let \( N = n(p + q) \) be even with \( n =: 2r \) even and \( p + q =: 2s + 1 \) odd. Then, in order for the \((m, m')\) 2-periodic trigonometric interpolation problem (1.5) to be regular on the \( 2n \) equidistant nodes \( \{x_k\}_{k=0}^{2n-1} \) in \([0, 2\pi)\), it is necessary and sufficient that

\[
(5.8) \quad \Delta(\alpha_j) \neq 0 \quad (\alpha_j = \frac{j - r}{2r}, \ j = 1, 2, \ldots, 2r),
\]

and that

\[
(5.9) \quad \Delta_0 \neq 0 \quad (\alpha_0 = -\frac{1}{2}, \ j = 0).
\]

Next, since \( \alpha_r = 0 \) from (5.8) and since \( \Delta(0) \) in this section is the same as that in §4 (except for the value of \( n \)), we immediately obtain the results of (4.14) and (4.16), which directly give the first part of the following analogue of Proposition 2 of section 4.

**Proposition 4.** Let \( N = n(p + q) \) be even with \( n =: 2r \) even and \( p + q =: 2s + 1 \) odd. Then, a necessary and sufficient condition that \( \Delta(0) \neq 0 \) is that

\[
(5.10) \quad E + E' - 1 = s = O + O'.
\]

Moreover, a necessary and sufficient condition that \( \Delta_0 \neq 0 \) is that either

\[
(5.11) \quad \begin{cases} 
E + O' = s + 1 & \text{and} \quad E' + O = s & \text{with} \quad \varepsilon = 0, \text{ or} \\
E + O' = s & \text{and} \quad E' + O = s + 1 & \text{with} \quad \varepsilon = 1.
\end{cases}
\]

For the proof of the last part of Proposition 4, recalling from (5.5) that \( \alpha_0 = -1/2 \), we may assume (without loss of generality) that the components of \( m \) and \( m' \) are ordered as in (4.17). Then, up to a nonzero multiplicative factor, the determinant \( \Delta_0 \) from (5.4) and (5.7),
of order $2s + 2$, becomes

$$
(5.12) \quad \Delta_0 = \begin{vmatrix}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\
(2s + 1)^{m_2} & 3^{m_2} & 1 & 1 & 3^{m_2} & (2s + 1)^{m_2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(2s + 1)^{m_{\ell}} & \cdots & 3^{m_{\ell}} & 1 & 1 & 3^{m_{\ell}} & \cdots & (2s + 1)^{m_{\ell}} \\
-(2s + 1)^{m_{\ell+1}} & \cdots & -3^{m_{\ell+1}} & -1 & 1 & 3^{m_{\ell+1}} & \cdots & (2s + 1)^{m_{\ell+1}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-(2s + 1)^{m_p} & \cdots & -3^{m_p} & -1 & 1 & 3^{m_p} & \cdots & (2s + 1)^{m_p} \\
-(2s + 1)^{m_1} & \cdots & (-1)^s 3^{m_1} & (-1)^{s+1} & (-1)^s & (-1)^{s+1} 3^{m_1} & \cdots & (2s + 1)^{m_1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-(2s + 1)^{m_k} & \cdots & (-1)^s 3^{m_k} & (-1)^{s+1} & (-1)^s & (-1)^{s+1} 3^{m_k} & \cdots & (2s + 1)^{m_k} \\
(2s + 1)^{m_{k+1}} & \cdots & (-1)^{s+1} 3^{m_{k+1}} & (-1)^s & (-1)^s & (-1)^{s+1} 3^{m_{k+1}} & \cdots & (2s + 1)^{m_{k+1}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(2s + 1)^{m_q} & \cdots & (-1)^{s+1} 3^{m_q} & (-1)^s & (-1)^s & (-1)^{s+1} 3^{m_q} & \cdots & (2s + 1)^{m_q} \\
1 & \cdots & 0 & 0 & 0 & 0 & \cdots & (-1)^{1+s}
\end{vmatrix}
$$

It is evident that the first $\ell$ rows, involving the exponents $m_1, \ldots, m_{\ell}$, and the $q - k$ rows, involving the exponents $m'_{k+1}, \ldots, m'_{q}$, of the determinant $\Delta_0$ of (5.12) are symmetric about its vertical center, while its remaining interior rows are antisymmetric about its vertical center. Thus, if we subtract the first column of $\Delta_0$ of (5.12) from its last $(2s + 2)$-nd column, its second column from its $(2s + 1)$-st column, etc., thereby forming $\hat{\Delta}_0$ (where $\hat{\Delta}_0 = \Delta_0$), then the last $s + 1$ columns of this determinant have at least $E + O'$ zero rows. If we assume that

$$
E + O' \geq s + 2,
$$

then applying the Laplace expansion to this $\hat{\Delta}_0$ about its last $s + 1$ columns shows that $\hat{\Delta}_0 = 0$. Thus, a necessary condition that $\Delta_0 \neq 0$ is that

$$
(5.13) \quad E + O' \leq s + 1,
$$

and, in a completely analogous manner, it similarly follows (on adding appropriate columns of $\Delta_0$) that a necessary condition that $\Delta_0 \neq 0$ is also that

$$
(5.14) \quad E' + O \leq s + 1.
$$
However, since \( E + E' + O + O' = 2s + 1 \) from (5.1), then equality cannot simultaneously hold in (5.13) and (5.14). Hence, either \( E + O' = s + 1 \) and \( E' + O = s \), or \( E + O' = s \) and \( E' + O = s + 1 \).

If \( E + O' = s + 1 \) and \( E' + O = s \), then the determinant \( \hat{\Delta}_0 \), formed above by subtracting the first column of \( \Delta_0 \) from its last column, etc., has the last \( s + 1 \) entries of its final row given by
\[
(0, 0, \cdots, 0, -1 + (-1)^{1+\varepsilon}).
\]

Clearly, the choice \( \varepsilon = 1 \) would make a zero row in (5.15) above, which, by the argument above, would result in \( \hat{\Delta}_0 = 0 \). Thus, in this first case of (5.11), a necessary condition that \( \hat{\Delta}_0 \neq 0 \) is that \( \varepsilon = 0 \).

We now show that the choice \( \varepsilon = 0 \), when \( E + O' = s + 1 \) and \( E' + O = s \), is a sufficient condition that \( \hat{\Delta}_0 \neq 0 \). For the determinant \( \hat{\Delta}_0 \), of order \( 2s + 2 \), constructed above, we have that the last \( s + 1 \) columns of \( \hat{\Delta}_0 \) has exactly \( s + 1 \) zero rows. Thus, in the Laplace expansion of \( \hat{\Delta}_0 \) in terms of its last \( s + 1 \) columns, there is at most one term which has to be examined, namely the product to two subdeterminants, each of order \( s + 1 \), where the first subdeterminant is taken from the last \( s + 1 \) columns of \( \hat{\Delta}_0 \) and rows corresponding to the integers of \( O \) and \( E' \) (where \( E' + O = s \)) and the final row of \( \hat{\Delta}_0 \) (which is zero from (5.15) except for its final entry which is \(-1\), since \( \varepsilon = 0 \)), and the second subdeterminant is complementary to the first subdeterminant, corresponding to the first \( s + 1 \) columns and the rows corresponding to the integers of \( E \) and \( O' \) (where \( E' + O = s + 1 \)). But on applying Lemma 2, it is readily seen that neither of these subdeterminants is zero; whence, \( \hat{\Delta}_0 \neq 0 \). This establishes the first necessary and sufficient condition of (5.11). We remark that in the same manner (on adding columns in (5.12)), one similarly obtains the second necessary and sufficient condition of (5.11).

To complete the proof of Theorem in the case (2.3i), we note that the proof of §4 shows that the assumptions, that the components of \( \mathbf{m} \) and \( \mathbf{m}' \) satisfy (5.10) and the alternation condition of (2.2), give that \( \Delta(\alpha) \neq 0 \) for any \( 0 < |\alpha| \leq 1/2 \). Thus, it remains only to show
that (2.5i) of Theorem 2 is valid, i.e., \( \varepsilon = 1 \) if \( p \) is even, and \( \varepsilon = 0 \) if \( p \) is odd. First, assume that \( p \) is even, say \( p := 2\ell \). Then, \( q := 2t + 1 \) is odd since \( p + q = 2s + 1 \) is odd. In this case, \( s := \ell + t \). Because \( p = 2\ell \) is even and because of the alternation condition (2.2), it necessarily follows that \( E = \ell = O \). With \( q = 2t + 1 \), assume that \( E' = t + 1 \) and if \( O' = t \). Then, \( E + E' - 1 = s = O + O' \) which satisfies (5.10), and \( E + O' = s \) and \( E' + O = s + 1 \). Thus from (5.11), \( \varepsilon = 1 \). Again with \( q = 2\ell + 1 \), assume on the other hand that \( E' = t \) and \( O' = t + 1 \). But then, \( E + E' - 1 = s - 1 \neq O + O' = s + 1 \), which violates the condition (5.10). Hence, when \( p \) is even, we can only satisfy the conditions of (5.10) and (5.11), with the latter case of (5.11) so that \( \varepsilon = 1 \), which gives the first part of (2.5i). The case when \( p \) is odd is completely similar and is omitted.

Finally, because the remaining cases ii) and iii) of Theorem 2 follow along the lines shown above, we leave these verifications to the reader.

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