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LEVEL SETS OF REAL ENTIRE FUNCTIONS AND THE LAGUERRE INEQUALITIES

George Csordas, Wayne Smith and **Richard S. Varga**[1]

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ABSTRACT: The level set structure of a real entire function f is investigated. The results establish a connection between the level sets of f, the Laguerre expression for f and the distribution of zeros of f. An application to the Riemann Hypothesis is also given.

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1. Introduction.

Let $S(A)$ denote the closed strip of width $2A$ in the complex plane $\mathbb{C}$ symmetric about the real axis:

$$S(A) = \{z \in \mathbb{C} \mid |\mathrm{Im}\,(z)| \leq A\},$$

where $A \geq 0$.

DEFINITION. *Let A be such that $0 \leq A < \infty$. We say that a real entire function f belongs to the class $\mathfrak{S}(A)$ if f is of the form*

$$f(z) = Ce^{-az^2+bz} z^m \prod_{k=1}^{\infty} (1 - z/z_k) e^{z/z_k}, \tag{1.1}$$

where $a \geq 0$, $z_k \in S(A) \setminus \{0\}$, and $\sum_{k=1}^{\infty} 1/|z_k|^2 < \infty$.

We allow functions in $\mathfrak{S}(A)$ to have only finitely many zeros by letting, as usual, $z_k = \infty$ and $0 = 1/z_k$, $k \geq k_0$, so that the canonical product in (1.1) is a finite product. If a function $f \in \mathfrak{S}(A)$, for some $A \geq 0$, and moreover f has only real zeros (i.e., if $A = 0$), then f is said to belong to the *Laguerre-Pólya class*, and we write $f \in \mathcal{L} - \mathcal{P}$. We say

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$f \in \mathcal{L}-\mathcal{P}^*$ if $f = pg$, where $g \in \mathcal{L}-\mathcal{P}$ and p is a real polynomial. Thus, $f \in \mathcal{L}-\mathcal{P}^*$ if and only if $f \in \mathfrak{S}(A)$, for some $A \geq 0$, and f has at most finitely many nonreal zeros.

We use $L[g]$ to denote the Laguerre expression associated with an entire function g; that is, we let

$$L[g] = (g')^2 - gg''.$$

Observe that if

$$p(z) = C\prod_{k=1}^{n}(z-a_k)$$

is a nonconstant real polynomial with only real zeros, then

$$L[p](x) = -p^2(x)\left(\frac{p'}{p}\right)'(x) = p^2(x)\sum_{k=1}^{n}\frac{1}{(x-a_k)^2} \geq 0, \tag{1.2}$$

for all $x \in \mathbb{R}$, with equality holding throughout if and only if x is a multiple zero of p.

A function g is in $\mathcal{L}-\mathcal{P}$ if and only if g is the uniform limit on compact subsets of $\mathbb{C}$ of real polynomials having only real zeros (see [16, Chapter VIII], [22] or [25]). Thus it follows that if $g \in \mathcal{L}-\mathcal{P}$, then $L[g](x) \geq 0$ for all $x \in \mathbb{R}$. Moreover, a calculation, similar to the one carried out for p in (1.2), shows that if $g \in \mathcal{L}-\mathcal{P}$ is not of the form $g(x) = C\exp(bx)$, then $L[g](x_0) = 0$ if and only if x_0 is a multiple zero of g. Our main goal in this paper is to investigate possible converses to this observation. That is, we wish to find conditions on the Laguerre expression of a function in $\mathfrak{S}(A)$ that imply that the function has only real zeros. A short computation shows that for $g(z) = z(z^2+1) \in \mathfrak{S}(1)$, $L[g](x) = 3x^4+1$, and so some hypotheses beyond $L[g](x) > 0$ are required if one hopes to conclude that g has only real zeros. The additional hypotheses we use involve the following family of functions closely related to g.

For an entire function g and for $\mu \in \mathbb{R}$, we define the function g_μ by the equation

$$g_\mu(x) = g(x+i\mu) + g(x-i\mu).$$

(We note that if $g \in \mathfrak{S}(A)$, then $g_\mu = 2\cos(\mu D)g$, where D denotes differentiation with respect to z (see [10]). All of our results involving the functions g_μ can be restated using these differential operators.) Thus if g is a real entire function, then g_μ is also a real entire function, and the relationship between the zero set of g_μ and that of g has been studied by several authors (see [2], [13], [15], [20], [22] and [24]). For example, it is well known

([2]) that if $g \in \mathfrak{S}(A)$, for some $A \geq 0$, then $g_\mu \in \mathcal{L}-\mathcal{P}$ for all real μ such that $|\mu| \geq A$. (It should be pointed out, however, that g_μ may be a constant function, even if g is not. The function $g(x) = \exp(x)$ and $\mu = \pi/2$ is an example.) In particular, if $g \in \mathcal{L}-\mathcal{P}$, then $g_\mu \in \mathcal{L}-\mathcal{P}$ for all $\mu \in \mathbb{R}$. Hence, if $g \in \mathcal{L}-\mathcal{P}$, then $L[g_\mu](x) \geq 0$ for all $x \in \mathbb{R}$ and all $\mu \in \mathbb{R}$. Moreover, it is not difficult to prove the following refinement of this implication. We use the notation $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$.

THEOREM I. *If $f \in \mathcal{L}-\mathcal{P}$ and if f is not of the form $C\exp(bz)$, then $L[f_\mu](x) > 0$ for all $\mu \neq 0$ and for all $x \in \mathbb{R}$.*

Proof. From the discussion preceding the statement of the theorem, it is sufficient to show that for any $\mu \neq 0$, f_μ has only simple zeros. Since $f \in \mathcal{L}-\mathcal{P}$ is of the form (1.1), with $z_k \in \mathbb{R}$, and f is not of the form $C\exp(bz)$,

$$(1.3) \qquad \operatorname{Im}\frac{\overline{f(z)}f'(z)}{|f(z)|^2} = \operatorname{Im}\frac{f'}{f}(z) = \operatorname{Im}\left(-2az + b + \frac{m}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{(z-z_k)} + \frac{1}{z_k}\right)\right) < 0,$$

for all $z \in \mathbb{H}^+$, as is readily verified. On the other hand, if x_0 is a multiple zero of f_μ, so that $f_\mu(x_0) = f'_\mu(x_0) = 0$, then $f(x_0+i\mu) = -f(x_0-i\mu)$ and $f'(x_0-i\mu) = -f'(x_0+i\mu)$. Since f is a real entire function, it follows that $\operatorname{Im}(f(x_0-i\mu)f'(x_0+i\mu)) = 0$. This contradicts (1.3) in the case that $\mu > 0$, and since $f_\mu = f_{-\mu}$ this completes the proof.

The main results in this paper are converses to Theorem I in which various additional assumptions are made about f. We do not know if the converse to Theorem I is valid in the absence of additional assumptions. The proofs of our main results involve the study of the level sets of f, that is, the sets

$$(1.4) \qquad \{z \in \mathbb{C} \mid \operatorname{Re}(e^{i\theta}f(z)) = 0\}, \quad \theta \in \mathbb{R}.$$

The connections between the Laguerre expression of f_μ, the level set $\operatorname{Re} f = 0$ and the zero set of f_μ are fundamental throughout this paper. The key observation is that for a real entire function f, $f_\mu(x) = 0$ if and only if $\operatorname{Re} f(x+i\mu) = 0$. To make this introduction easier to read, we only state the special cases of our results that correspond to $\theta = 0$ in (1.4). They will be stated and proved in their full generality in later sections.

The next theorem relates the Laguerre expressions associated with f_μ to the mapping properties of the logarithmic derivative of f. A corollary is the validity of the converse to Theorem I provided that f has at least one zero and at most finitely many nonreal zeros.

THEOREM II. *If f is a real entire function and if $L[f_\mu](x) > 0$, for all $\mu \neq 0$ and for all $x \in \mathbb{R}$, then either the set*

$$\Lambda = \{z \in \mathbb{H}^+ \mid Im \frac{f'}{f}(z) > 0\}$$

is empty and $f \in \mathcal{L}-\mathcal{P}$, or all components of Λ are unbounded.

COROLLARY I. *Suppose that $f \in \mathcal{L}-\mathcal{P}^*$ and f is not of the form $C\exp(bz)$. Then $f \in \mathcal{L}-\mathcal{P}$ if and only if $L[f_\mu](x) > 0$ for all $\mu \neq 0$ and for all $x \in \mathbb{R}$.*

Remark. The strict inequality $L[f_\mu](x) > 0$ in the statement of Corollary I is necessary. This can be seen by considering the polynomial $f(z) = z(z^2+1)$, which belongs to $\mathcal{L}-\mathcal{P}^*$ but not to $\mathcal{L}-\mathcal{P}$, since f has nonreal zeros. A computation shows that $L[f_\mu](x) = 4(3x^4 + (3\mu^2-1)^2) \geq 0$ for all $x, \mu \in \mathbb{R}$.

In our next result, we do not assume that f has only finitely many nonreal zeros. However, we must make certain assumptions concerning the density of the real zeros of f. For $\Omega \subset \mathbb{C}$ we define $Z(f;\Omega)$ to be the number of zeros (counting multiplicities) of f that are in Ω, $Z_R(f;\Omega)$ to be the number of zeros of f in $\Omega \cap \mathbb{R}$ and $Z_C(f;\Omega) = Z(f;\Omega) - Z_R(f;\Omega)$. We also define

$$d(f;x) = \frac{Z_R(f;\{0 \leq \mathrm{Re}\,(z) \leq x\})}{Z(f;\{0 \leq \mathrm{Re}\,(z) \leq x\})}, \tag{1.5}$$

so that $d(f;x)$ measures the density of real zeros of f in $\{z \mid 0 \leq \mathrm{Re}\,(z) \leq x\}$.

THEOREM III. *Let $f \in \mathfrak{S}(A)$. Suppose that there exists $\delta > 0$ such that both*

$$\min\{d(f(z);x),\, d(f(-z);x)\} \geq \delta$$

and

$$\min\{Z(f;\{0 \leq Re(z) \leq x\}),\, Z(f;\{-x \leq Re(z) \leq 0\})\} \geq \delta x$$

hold for x sufficiently large. Then $f \in \mathcal{L}-\mathcal{P}$ if and only if $L[f_\mu](x) > 0$ for all $\mu \neq 0$ and for all $x \in \mathbb{R}$.

The required facts concerning the level sets (1.4) are established in §2, and then more general versions of Theorem I, Theorem II and Corollary I involving the parameter θ are proved (see Theorem 2.1, Theorem 2.5 and Corollary 2.6, respectively). The proof of the formulation of Theorem III involving θ can be found in §3 (see Theorem 3.5). The theorems stated above correspond to the choice $\theta = 0$.

In §4 we consider applications of our results to the Riemann Hypothesis. Here we obtain new conditions that are both necessary and sufficient for the Riemann Hypothesis to be valid.

2. Level sets of real entire functions.

The purpose of this section is to establish some results relating the geometric nature of the level set structure of a real entire function f to the distribution of its zeros . In particular, we will establish a relationship between the components of the set

$$\Lambda(f) = \{z \in \mathbb{H}^+ \mid \operatorname{Im} \frac{f'}{f}(z) > 0\}$$

when $\Lambda(f)$ is not empty and the Laguerre inequalities $L[f_\mu](x) \geq 0$, $\mu, x \in \mathbb{R}$, hold.

In our preliminary lemmas we will make use of several fundamental, although elementary, properties of level curves which we briefly review here for the reader's convenience. We first recall that if f is a nonconstant real entire function, then, for each fixed $\theta \in \mathbb{R}$, a component, γ, of the level set

$$\{z \in \mathbb{C} \mid \operatorname{Re}(e^{i\theta} f(z)) = 0\} \tag{2.1}$$

is a piecewise analytic curve. Such a curve γ is called a *level curve* of f. Recall that by the local mapping properties of an analytic function, every zero of f is on some level curve of f. If z_0 is a critical point of f, that is, if $f'(z_0) = 0$, and if z_0 is on the level curve γ, then z_0 is said to be a *branch point* of γ.

Now, if γ has no branch points, then it follows from the local mapping properties of f that the restriction of f to γ is locally a homeomorphism. Moreover, if $\gamma(z_0, z_1)$ is the portion of γ between distinct points z_0, z_1 with $f(z_0) = f(z_1)$, then since f is nonconstant, $|f|$ must achieve on the compact set $\gamma(z_0, z_1)$ a maximum and a minimum, one of which is distinct from $|f(z_0)|$. It is easy to see that the derivative of f must vanish at the point that this extreme value is achieved. Thus, if γ has no branch points, then f is one-to-one on γ, and the restriction of f to γ is a homeomorphism. Hence in this case f can have at most one zero on γ. (Simple examples show that f need not have a zero on γ.) Since f is nonconstant, by the maximum principle γ cannot be a closed bounded curve. Moreover, if γ has no branch points, then γ is an analytic curve which separates the plane.

For a real entire function f and $\mu, \theta \in \mathbb{R}$, define

$$f_{\mu,\theta}(x) = e^{i\theta} f(x + i\mu) + e^{-i\theta} f(x - i\mu).$$

It is clear that $f_{\mu,\theta}$ is a real entire function, and it is well known (see [2, Theorem 8]) that if $f \in \mathfrak{S}(A)$, then $f_{\mu,\theta} \in \mathfrak{S}(A)$. Moreover, if $\mu \geq A$, then $f_{\mu,\theta} \in \mathcal{L}-\mathcal{P}$. We can now give our generalization of Theorem I. The proof of this generalization, in contrast to that of Theorem I, is geometric in nature and is based on the structure of the level set

$$R_{f,\theta} = \{z \in \mathbb{C} \mid \mathrm{Re}\,(e^{i\theta} f(z)) = 0\}.$$

The connection between the Laguerre expression of $f_{\mu,\theta}$, the level set $R_{f,\theta}$ and the zero set of $f_{\mu,\theta}$ is fundamental throughout this paper. The key observation is that for a real entire function f, $f_{\mu,\theta}(x) = 0$ if and only if $x + i\mu \in R_{f,\theta}$.

THEOREM 2.1. *Let $\theta \in \mathbb{R}$ be arbitrary, and suppose that $f \in \mathfrak{S}(A)$. If f is not of the form $C\exp(bz)$, then $L[f_{\mu,\theta}](x) > 0$ for all $\mu > A$ and for all $x \in \mathbb{R}$.*

Proof. Since $f_{\mu,\theta} \in \mathcal{L}-\mathcal{P}$ for $\mu > A$, we see by the background material in the Introduction that it is sufficient to prove that, for each $\mu > A$, $f_{\mu,\theta}$ has only simple zeros. Suppose to the contrary that $\mu > A$ and x_0 is a multiple zero of $f_{\mu,\theta}$. Thus, if $\eta > 0$ and $|\lambda - \mu|$ is sufficiently small, then $f_{\lambda,\theta}$ has at least two zeros in the interval $(x_0 - \eta, x_0 + \eta)$. It follows that the horizontal line $\mathrm{Im}\,(z) = \mu$ intersects $R_{f,\theta} \cap \{z \in \mathbb{C} \mid |Re(z) - x_0| < \delta\}$ just at the point $x_0 + i\mu$ if $\delta > 0$ is sufficiently small, while the line $\mathrm{Im}\,(z) = \lambda$ intersects this same set in at least two points if $0 < |\lambda - \mu| < \epsilon$ and $\epsilon > 0$ is sufficiently small. Thus the set $R_{f,\theta}$ contains at least two analytic curves that intersect at $x_0 + i\mu$, which happens if and only if $f'(x_0 + i\mu) = 0$. Hence, f' has a zero outside of the set $S(A)$, which contradicts the well known fact that $\mathfrak{S}(A)$ is closed under differentiation [2].

Remark. A proof of Theorem 2.1 could also be based on the Hermite-Biehler theory (see [16, Chapter VII]), using an argument which is *mutatis mutandis* the same as the one used in the proof of Theorem I in the Introduction.

The following lemma shows that if a level curve γ (in $\mathbb{H}^+$) of a real entire function f has either a branch point or a horizontal tangent line, then the Laguerre expression $L[f_{\mu,\theta}](x)$ must be nonpositive for some $x, \mu \in \mathbb{R}$, $\mu \neq 0$.

__LEMMA 2.2.__ Let f be a real entire function, let θ be any real number, and let γ be a component of the set

$$\{z \in \mathbb{H}^+ \mid \operatorname{Re}(e^{i\theta} f(z)) = 0\}.$$

If γ has a branch point or if there is a point on γ where γ has a horizontal tangent line, then

$$L[f_{\mu_1,\theta}](x_1) = 0,$$

for some $x_1, \mu_1 \in \mathbb{R}$, $\mu_1 > 0$.

Proof. Suppose first that γ has a branch point, i.e., there is a point $z_1 = x_1 + i\mu_1$ on γ where $f'(z_1) = 0$. Then, $f_{\mu_1,\theta}(x_1) = 2\operatorname{Re}(e^{i\theta} f(x_1 + i\mu_1)) = 0$ and $f'_{\mu_1,\theta}(x_1) = 2\operatorname{Re}(e^{i\theta} f'(x_1 + i\mu_1)) = 0$. Consequently, in this case $L[f_{\mu_1,\theta}](x_1) = 0$.

Next suppose that γ has a horizontal tangent line at $z_1 = x_1 + iy_1$. By the preceding argument, we may assume that z_1 is not a branch point of γ. Let $z(t)$ be the analytic parameterization of γ in a neighborhood of z_1 such that $f(z(t)) = ie^{-i\theta}t$. Differentiation of this equation with respect to t yields that $f'(z(t))z'(t) = ie^{-i\theta}$. Let $t_1 \in \mathbb{R}$ be such that $z(t_1) = z_1$, and observe that $z'(t_1)$ is real and nonzero. Hence $\operatorname{Re}(e^{i\theta} f'(z(t_1))) = 0$, so $f'_{y_1,\theta}(x_1) = 2\operatorname{Re}(e^{i\theta} f'(z_1)) = 0$. Since we also have $f_{y_1,\theta}(x_1) = 2\operatorname{Re}(e^{i\theta} f(z_1)) = 0$, we get that $L[f_{y_1,\theta}](x_1) = 0$ and the proof is complete.

__LEMMA 2.3.__ *Let f be a real entire function and let θ be any real number. Suppose that*

$$\Lambda = \{z \in \mathbb{H}^+ \mid Im \frac{f'}{f}(z) > 0\} \neq \emptyset \tag{2.2}$$

and that V is a bounded component of Λ. Then there is at least one component γ of the level set

$$\{z \in \mathbb{H}^+ \mid Re(e^{i\theta} f(z)) = 0\} \tag{2.3}$$

which intersects the boundary, ∂V, *of* V *at a point in* $\mathbb{H}^+$. *A point of intersection in* $\mathbb{H}^+$ *of* γ *and* ∂V *is either (i) a zero of* f, *(ii) a critical point of* f, *or (iii) a point where* γ *has a horizontal tangent line. Moreover, if* γ *has no branch points, then there exists at least one point* $z_1 \in \partial V \cap \gamma$ *where* γ *has a horizontal tangent line.*

Proof. Since $\Lambda \neq \emptyset$, f is nonconstant and whence it follows by the maximum principle that f has at least one zero $z_0 \in \partial V$. Clearly, z_0 is a simple pole of f'/f with positive residue, and so $\mathrm{Im}\,(z_0) > 0$. Indeed, if z_1 is a real simple pole of f'/f (with necessarily positive residue), then

$$\mathrm{Im}\,\frac{f'}{f}(z) < 0$$

at all points $z \in \mathbb{H}^+$ sufficiently close to z_1, and so $z_1 \notin \partial V$. Next, recall (by the remarks at the beginning of this section) that z_0, being a zero of f, is on at least one level curve γ, where γ is a component of the level set (2.3). Now consider a point $z_2 \in \partial V \cap \gamma$ that is neither a critical point nor a zero of f. As in the proof of Lemma 2.2, let $z(t)$ be the analytic parameterization of γ in a neighborhood of z_2 such that $f(z(t)) = ie^{-i\theta}t$. Logarithmic differentiation with respect to t yields that

$$\frac{f'}{f}(z(t))z'(t) = \frac{1}{t}\,. \tag{2.4}$$

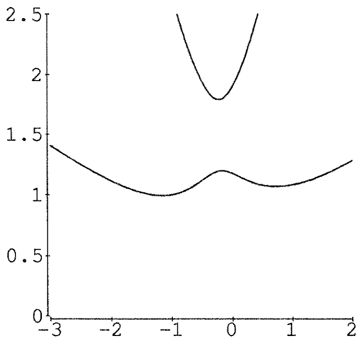
Let t_2 be such that $z(t_2) = z_2$, and observe that since $z_2 \in \partial V \cap \mathbb{H}^+$, $\frac{f'}{f}(z_2)$ is real and hence from (2.4) we conclude that $\mathrm{Im}\,(z'(t_2)) = 0$ and $\mathrm{Re}\,(z'(t_2)) \neq 0$. Consequently, γ has a horizontal tangent line at z_2.

Finally, to prove the last assertion of the lemma, consider a level curve γ which has no branch points and which intersects $\partial V \cap \mathbb{H}^+$ at z_0. If γ has a horizontal tangent at z_0, there is nothing more to prove. So assume γ does not have a horizontal tangent at z_0, and observe that, by the part of the lemma already established, z_0 is a zero of f. As noted earlier, z_0 is a simple pole of f'/f with positive residue, and so the horizontal line $\mathrm{Im}\,(z) = \mathrm{Im}\,(z_0)$ is tangent to ∂V at z_0. From this it follows that $\gamma \cap V \neq \emptyset$. Since V is bounded and γ is unbounded, γ must intersect ∂V at a point z_1, $z_1 \neq z_0$. Since γ has no branch points, f is one-to-one on γ and so f cannot vanish at z_1. Also, if θ is not an odd multiple of $\pi/2$, then γ can intersect $\mathbb{R}$ only at a zero of f, so $\mathrm{Im}\,(z_1) > 0$. Therefore, by the previously established part of the lemma (cf. (iii)), γ has a horizontal tangent line at

z_1. On the other hand, if θ is an odd multiple of $\pi/2$, then $\mathbb{R}$ is contained in the level set (2.3). Thus the assumption that γ has no branch points implies that γ can not intersect $\mathbb{R}$, and the argument is completed as before. This completes the proof of the lemma.

Remark. Lemma 2.3 should be compared to the following result of Sheil-Small [30, Lemma 1]: If a real entire function f has only real zeros, then the components of Λ (cf. (2.2)) are all unbounded.

We now present an example to illustrate Lemma 2.2 and Lemma 2.3, and the geometric ideas that motivate their use in this paper. In this example we just consider the level sets corresponding to $\theta = 0$. Consider the function $f(z) = 5 + 3z + 5z^2 + z^3 + z^4$, with zeros at $-.404\ldots \pm 1.152i\ldots$ and $-.0953\ldots \pm 1.829i\ldots$.



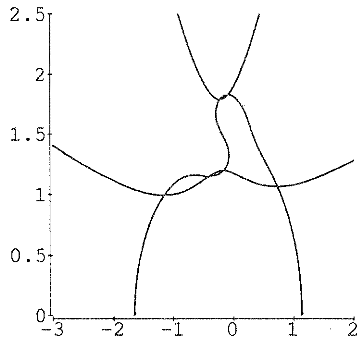


FIGURE 1. The level set $\mathrm{Re}\,(f) = 0$, first alone and then with the level set $\mathrm{Im}\,\frac{f'}{f} = 0$, for the function $f(z) = 5 + 3z + 5z^2 + z^3 + z^4$.

The level sets $\mathrm{Re}\,(f) = 0$ and $\mathrm{Im}\,\frac{f'}{f} = 0$ intersect at the zeros of f as well as at each point where $\mathrm{Re}\,(f) = 0$ has a horizontal tangent line. The level set $\mathrm{Im}\,\frac{f'}{f} = 0$ also has a horizontal tangent line at each zero of f. Furthermore, since the real zeros of f_μ occur at the points of intersection of the line $y = \mu$ with $\mathrm{Re}\,(f) = 0$, we can easily determine the number of real zeros of f_μ by studying Figure 1. In particular, using the notation $Z_R(g)$ for the number of real zeros of a function g, we see how Figure 1 reflects that $Z_R(f_{.5}) = 0$, $Z_R(f_1) = 2$, $Z_R(f_{1.1}) = 4$, $Z_R(f_{1.5}) = 2$ and $Z_R(f_\mu) = 4$ for $\mu \geq 1.789\ldots$. This should be compared to the observation made earlier that f_μ has only real zeros for $\mu \geq 1.829\ldots$, since the zeros of f belong to $S(1.829\ldots)$. We also note that this example shows that it is possible to have $g_{\mu_1} \in \mathcal{L}-\mathcal{P}$ but $g_{\mu_2} \notin \mathcal{L}-\mathcal{P}$, for some $\mu_2 > \mu_1$.

In the proof of the main theorem of this section (Theorem 2.5 below) we will make use of the following known characterization of functions in the Laguerre-Pólya class (see [30, Lemma 4] and for special cases [13] and [5, Theorem 2.12]).

LEMMA 2.4. *Let f be a real entire function. Then $f \in \mathcal{L}-\mathcal{P}$ if and only if $Im \frac{f'}{f}(z) \leq 0$ for all $z \in \mathbb{H}^+$. Moreover, if $f \in \mathcal{L}-\mathcal{P}$ and f is not of the form $C\exp(bz)$, then $Im \frac{f'}{f}(z) < 0$ for all $z \in \mathbb{H}^+$.*

THEOREM 2.5. *Let $\theta in \mathbb{R}$. If f is a real entire function and $L[f_{\mu,\theta}](x) > 0$, for all $\mu > 0$ and for all $x \in \mathbb{R}$, then either the set*

$$\Lambda = \{z \in \mathbb{H}^+ \mid Im \frac{f'}{f}(z) > 0\}$$

is empty and $f \in \mathcal{L}-\mathcal{P}$, or all the components of Λ are unbounded.

Proof. If $\Lambda = \emptyset$, then $f \in \mathcal{L}-\mathcal{P}$ by Lemma 2.4. Suppose $\Lambda \neq \emptyset$ and V is a bounded component of Λ. Then by Lemma 2.3 there is at least one component γ of the level set $\{z \in \mathbb{H}^+ \mid \mathrm{Re}\,(e^{i\theta} f(z)) = 0\}$ which intersects ∂V. Moreover, either γ has a branch point or γ intersects ∂V at a point where γ has a horizontal tangent line. In either case, by Lemma 2.2, $L[f_{\mu_1,\theta}](x_1) \leq 0$ for some $\mu_1, x_1 \in \mathbb{R}$, $\mu_1 > 0$. Hence, if $L[f_{\mu,\theta}](x) > 0$, for all $\mu > 0$ and $x \in \mathbb{R}$ and $\Lambda \neq \emptyset$, then all the components of Λ are unbounded.

We close this section with some applications of Theorem 2.5. Our first application provides a necessary and sufficient condition for a function $f \in \mathcal{L}-\mathcal{P}^*$ to have only real zeros.

COROLLARY 2.6. *Let θ be any real number and suppose that $f \in \mathcal{L}-\mathcal{P}^*$ and f is not of the form $C\exp(bz)$. Then $f \in \mathcal{L}-\mathcal{P}$ if and only if $L[f_{\mu,\theta}](x) > 0$ for all $\mu \neq 0$ and $x \in \mathbb{R}$.*

Proof. If $f \in \mathcal{L}-\mathcal{P}$ and f is not of the form $C\exp(bz)$, then $L[f_{\mu,\theta}](x) > 0$, for all $\mu \neq 0$ and $x \in \mathbb{R}$, by Theorem 2.1. Conversely, suppose that $f \in \mathcal{L}-\mathcal{P}^*$ and that $L[f_{\mu,\theta}](x) > 0$, for all $\mu \neq 0$ and $x \in \mathbb{R}$. By the definition of $\mathcal{L}-\mathcal{P}^*$, $f = pg$, where $g \in \mathcal{L}-\mathcal{P}$ and p is a real polynomial. Also, by Lemma 2.4, $\operatorname{Im} \frac{g'}{g}(z) \le 0$ for all $z \in \mathbb{H}^+$, and so

$$\Lambda = \{z \in \mathbb{H}^+ \mid \operatorname{Im} \frac{f'}{f}(z) > 0\} \subset \{z \in \mathbb{H}^+ \mid \operatorname{Im} \frac{p'}{p}(z) > 0\}.$$

If $z_k = x_k + iy_k$, $1 \le k \le n$, are the nonreal zeros of p in $\mathbb{H}^+$, then a short computation (see for example [15], [20], [22] or [27]) shows that

$$\{z \in \mathbb{H}^+ \mid \operatorname{Im} \frac{p'}{p}(z) > 0\} \subset \bigcup_{k=1}^{n} D(x_k, y_k), \tag{2.5}$$

where $D(a, r) = \{z \in \mathbb{C} \mid |z - a| \le r\}$. It follows that Λ is bounded. Theorem 2.5 now implies that $\Lambda = \emptyset$ and $f \in \mathcal{L} - \mathcal{P}$.

<u>**COROLLARY 2.7.**</u> *Let θ be any real number and let $\varphi : [0, R] \to \mathbb{R}$, where $0 < R < \infty$. Suppose that $\varphi \in C^1[0, R]$ and that $\varphi(R) \ne 0$. Define*

$$f(z) = \int_0^R \varphi(t) \cos(zt)\, dt.$$

Then $f \in \mathcal{L} - \mathcal{P}$ if and only if $L[f_{\mu,\theta}](x) > 0$ for all $\mu \ne 0$ and for all $x \in \mathbb{R}$.

Proof. For the reader's convenience, we first outline the proof of the known fact that $f \in \mathcal{L} - \mathcal{P}^*$ (see [11, p. 100, Theorem 3.4.56]). Note that f is a real entire function of exponential type. Integration by parts yields

$$zf(z) = \varphi(R) \sin(zR) - \int_0^R \varphi'(t) \sin(zt)\, dt.$$

Since $\varphi(R) \ne 0$ and φ' is continuous, it follows from the Riemann-Lebesgue Lemma that $f(z)$ has an infinite number of real zeros. Next, let

$$r_n(z) = 2 \sum_{k=0}^{n} \frac{(-1)^k (k + \frac{1}{2})\pi f\left(\frac{(k+\frac{1}{2})\pi}{R}\right)}{\left((k + \frac{1}{2})\pi)^2 - R^2 z^2\right)}.$$

Since the sequence

$$\left\{(-1)^k (k + \frac{1}{2})\pi f\left(\frac{(k + \frac{1}{2})\pi}{R}\right)\right\}_0^\infty$$

has at most a finite number of sign changes, we conclude that $r_n(z)$ has at most M nonreal zeros, where M is independent of n. But

$$r_n(z) \to \frac{f(z)}{\cos(zR)}$$

uniformly on compact subsets of $\mathbb{C}\backslash\left\{\frac{(k+\frac{1}{2})\pi}{R}\right\}_{-\infty}^{\infty}$ (see [28, Part III, Problem #166]). Thus, by Hurwitz' theorem, $f \in \mathcal{L}-\mathcal{P}^*$. Since f is not of the form $C\exp(bz)$, the conclusion of this corollary follows from Corollary 2.6.

Remark. If, for example, we assume that $\varphi \in C^2[0,R]$, $\varphi(t) > 0$, $\varphi'(t) < 0$ and $\varphi''(t) < 0$, for $t \in [0,R]$, then it is known (see [28, Part V, Problem #173]) that $f \in \mathcal{L}-\mathcal{P}$. For additional results of this type see [26].

We will provide some other applications of Theorem 2.5, involving Fourier transforms, in §4.

3. Functions with zeros satisfying a density condition.

In this section we establish that certain conditions, which do not include the assumption that f has only finitely many nonreal zeros, are sufficient for a real entire function f to be in $\mathcal{L}-\mathcal{P}$. For the proofs, however, we need to assume that the real zeros of f satisfy a density condition involving the function $d(f;x)$ defined in (1.5). We begin with an elementary lemma.

LEMMA 3.1. *If $f \in \mathfrak{S}(A)$ and $M > 0$ is any positive number, then*

$$\liminf_{x\to\infty} \frac{Z(f;\{x-M \le Re(z) \le x+M\})}{x} = 0.$$

Proof. Suppose there exist $\epsilon > 0$ and $M > 0$ such that

$$Z(f;\{2jM \le \mathrm{Re}(z) \le 2(j+1)M\}) > \epsilon 2jM, \qquad j \ge j_0.$$

But then

$$\sum_{k=1}^{\infty}\frac{1}{|z_k|^2} \ge \sum_{j=j_0}^{\infty}\frac{Z(f;\{2jM \le \mathrm{Re}(z) \le 2(j+1)M\})}{(2(j+1)M)^2} \ge \sum_{j=j_0}^{\infty}\frac{\epsilon 2jM}{(4Mj)^2} = \infty,$$

which contradicts the assumption that $f \in \mathfrak{S}(A)$.

The next lemma is the key to the results of this section.

LEMMA 3.2. *Let $f \in \mathfrak{S}(A_0)$. Suppose that, for some $A > 0$,*

$$w : [0,\infty) \to S(A) \cap \mathbb{H}^+$$

is a parameterization of a simple piecewise analytic curve σ such that $f(w(t)) \neq 0$, $t \geq 0$, and $\lim_{t\to\infty} Re(w(t)) = \infty$. Suppose there exist $\delta > 0$ and $C_0 > 0$ such that, for all x sufficiently large, both

$$d(f;x) \geq \delta, \tag{3.1}$$

and

$$Z(f;\{0 \leq Re(z) \leq x\}) \geq C_0 x, \tag{3.2}$$

where $d(f;x)$ was defined in (1.5). Then

$$\liminf_{s\to\infty} Im \int_{\sigma(0,s)} \frac{f'}{f}(z)\,dz = -\infty,$$

where $\sigma(0,s)$ denotes the portion of σ corresponding to $0 \leq t \leq s$.

Proof. Clearly we may assume without loss of generality that $a = 0$, $b = 0$ and $m = 0$ in (1.1). Choose a curve η so that the union of the traces $\sigma \cup \eta$ is a simple, piecewise analytic curve contained in $S(A) \cap \mathbb{H}^+$ that separates $2Ai$ and $-2Ai$. Denote the components of $\mathbb{C} \setminus (\sigma \cup \eta)$ containing these points by G^+ and G^- respectively.

Let $\epsilon > 0$ be a small number to be specified later, and let

$$M = \max\{A/\epsilon, 1/\epsilon - \mathrm{Re}(w(0))\}. \tag{3.3}$$

By Lemma 3.1 we can find a sequence of points $x_n \to \infty$ such that

$$Z(f;\{x_n - M \leq \mathrm{Re}(z) \leq x_n + M\}) < \epsilon x_n, \qquad n \geq 1. \tag{3.4}$$

Since $x_n \to \infty$ we may also assume that $x_n \geq 2M + \mathrm{Re}(w(0))$ and $d(f;x_n) \geq \delta$ for all n, by (3.1).

For each n, we factor f as $f = f_1 f_2 f_3$, where

$$f_1(z) = \prod_{\mathrm{Re}(w(0))+M \leq \mathrm{Re}(z_k) \leq x_n - M} (1 - z/z_k)e^{z/z_k}$$

and

$$f_2(z) = \prod_{\mathrm{Re}(w(0))-M \leq \mathrm{Re}(z_k) < \mathrm{Re}(w(0))+M} (1 - z/z_k)e^{z/z_k} \prod_{x_n - M < \mathrm{Re}(z_k) \leq x_n + M} (1 - z/z_k)e^{z/z_k}.$$

For each $x > \operatorname{Re}(w(0))$, we can find $t(x) > 0$ such that $\operatorname{Re}(w(t(x))) = x$ and if $s > t(x)$, then $\operatorname{Re}(w(s)) > x$. Let $t_n = t(x_n)$. We consider the integral, over $\sigma(0, t_n)$, of the logarithmic derivative of each f_j separately.

Let Γ_n^- denote the polygonal path $[w(0), \operatorname{Re}(w(0)), x_n, w(t_n)]$ and let Γ_n^+ denote the polygonal path $[w(0), \operatorname{Re}(w(0)) + i, x_n + i, w(t_n)]$. For $n \geq 1$ and $k \geq 1$, let

$$I(z_k, n) = \operatorname{Im} \int\limits_{\sigma(0,t_n)} \left(\frac{1}{z - z_k} + \frac{1}{z - \overline{z_k}} + \frac{1}{z_k} + \frac{1}{\overline{z_k}}\right) dz.$$

Consider a zero z_k of f satisfying

$$\operatorname{Re}(w(0)) + M \leq \operatorname{Re}(z_k) \leq x_n - M, \tag{3.5}$$

and assume first that $z_k \in G^+$. The choice of t_j assures that the winding number

$$n(\Gamma_j^- - \sigma(0, t_j), z_k) = 0.$$

Thus, by Cauchy's theorem,

$$I(z_k, n) = \int\limits_{\Gamma_n^-} \left(\frac{1}{z - z_k} + \frac{1}{z - \overline{z_k}} + \frac{1}{z_k} + \frac{1}{\overline{z_k}}\right) dz.$$

By considering branches of the logarithms $\log(z - z_k)$ and $\log(z - \overline{z_k})$ on

$$\mathbb{C} \setminus \{z \mid \operatorname{Re}(z) = \operatorname{Re}(z_k) \text{ and } |\operatorname{Im}(z)| \geq \operatorname{Im}(z_k)\},$$

as a consequence of (3.3), (3.5) and the assumption that $|\operatorname{Im}(z_k)| \leq A_0$, we see that

$$\begin{aligned} &I(z_k, n) \\ &= \operatorname{Im} \log\left(\frac{w(t_n) - z_k}{w(0) - z_k}\right) + \operatorname{Im} \log\left(\frac{w(t_n) - \overline{z_k}}{w(0) - \overline{z_k}}\right) + 2\operatorname{Im}(w(t_n) - w(0))\frac{\operatorname{Re}(z_k)}{|z_k|^2} \\ &< (\pi + \epsilon O(1)) + (-\pi + \epsilon O(1)) + 2A\epsilon \\ &= \epsilon O(1). \end{aligned} \tag{3.6}$$

Here, and for the remainder of the proof, the positive bound implicit in $O(1)$ is independent of ϵ, k and n. Moreover, the notation $p(x) < O(1)$ just asserts the existence of an upper bound for $p(x)$, while $p(x) = O(1)$ is the statement that $p(x)$ is bounded. Thus, $-x^2 < O(1)$ and $-x^2 \neq O(1)$. If $z_k \in G^-$ satisfies (3.5) and $\operatorname{Im}(z_k) \neq 0$, then a similar argument, using Γ_n^+ in place of Γ_n^- and branches of the logarithms $\log(z-z_k)$ and $\log(z-\overline{z_k})$ on

$$\mathbb{C} \setminus \{z \mid \operatorname{Re}(z) = \operatorname{Re}(z_k) \text{ and } \operatorname{Im}(z) \leq \operatorname{Im}(z_k)\},$$

shows that

$$I(z_k, n) < (-\pi + \epsilon O(1)) + (-\pi + \epsilon O(1)) + 2A\epsilon < \epsilon O(1) \tag{3.7}$$

in this case as well.

Finally, if $\operatorname{Im}(z_k) = 0$ and z_k satisfies (3.5), set

$$J(z_k, n) = \operatorname{Im} \int_{\sigma(0,t_n)} \left(\frac{1}{z - z_k} + \frac{1}{z_k}\right) dz.$$

By considering a suitable branch of the logarithm $\log(z - z_k)$ on

$$\mathbb{C} \setminus \{z \mid \operatorname{Re}(z) = \operatorname{Re}(z_k) \text{ and } \operatorname{Im}(z) \leq 0\},$$

we have that

$$\begin{aligned} J(z_k, n) &= \operatorname{Im} \log\left(\frac{w(t_n) - z_k}{w(0) - z_k}\right) + \frac{1}{z_k}\operatorname{Im}(w(t_n) - w(0)) \\ &< -\pi + \epsilon O(1), \end{aligned} \tag{3.8}$$

by (3.3) and (3.5).

Combining (3.6), (3.7) and (3.8), it follows that

$$\begin{aligned} &\operatorname{Im} \int_{\sigma(0,t_n)} \frac{f_1'}{f_1}(z)\, dz \\ &\leq \sum_{\substack{\operatorname{Re}(w(0))+M \leq \operatorname{Re}(z_k) \leq x_n - M \\ Im z_k \neq 0}} I(z_k, n) + \sum_{\substack{\operatorname{Re}(w(0))+M \leq \operatorname{Re}(z_k) \leq x_n - M \\ Im z_k = 0}} J(z_k, n) \\ &< \epsilon O(1) Z_C(f; \{\operatorname{Re}(w(0)) + M \leq \operatorname{Re}(z_k) \leq x_n - M\}) \\ &\qquad + (-\pi + \epsilon O(1)) Z_R(f; \{\operatorname{Re}(w(0)) + M \leq \operatorname{Re}(z_k) \leq x_n - M\}) \\ &= Z(f; \{\operatorname{Re}(w(0)) + M \leq \operatorname{Re}(z_k) \leq x_n - M\}) \\ &\qquad \times [\epsilon O(1)(1 - d(f; x_n)) + (-\pi + \epsilon O(1)) d(f; x_n) + o(1)] \qquad (n \to \infty) \\ &\leq Z(f; \{\operatorname{Re}(w(0)) + M \leq \operatorname{Re}(z_k) \leq x_n - M\})[\epsilon O(1) - \pi\delta + o(1)] \\ &\leq -\delta C_0 x_n, \end{aligned} \tag{3.9}$$

provided that ϵ was chosen to be sufficiently small compared to δ and n is sufficiently large. In the last two inequalities, (3.1) and (3.2) were used.

We now consider f_2. For the zeros of f_2 satisfying $x_n - M \leq \text{Re}\,(z_k) \leq x_n + M$, we use the elementary estimates that

$$I(z_k, n) \leq 4\pi + \epsilon O(1) \leq O(1),$$

and similarly $J(z_k, n) \leq O(1)$. Since there are only finitely many other zeros of f_2, and they do not depend on n, we get that

$$\begin{aligned}
&\text{Im} \int_{\sigma(0,t_n)} \frac{f_2'}{f_2}(z)\,dz \\
(3.10)\qquad &\leq O(1) + \sum_{\substack{x_n - M \leq \text{Re}\,(z_k) \leq x_n + M \\ Im z_k \neq 0}} I(z_k, n) + \sum_{\substack{x_n - M \leq \text{Re}\,(z_k) \leq x_n + M \\ Im z_k = 0}} J(z_k, n) \\
&\leq (1 + \epsilon x_n) O(1),
\end{aligned}$$

by (3.4).

For f_3, by Cauchy's theorem and the fact that f_3 is real on $\mathbb{R}$,

$$\begin{aligned}
\text{Im} \int_{\sigma(0,t_n)} \frac{f_3'}{f_3}(z)\,dz &= \text{Im} \int_{\Gamma_n^-} \frac{f_3'}{f_3}(z)\,dz \\
(3.11)\qquad &= \text{Im} \int_{[w(0),\text{Re}\,(w(0))]} \frac{f_3'}{f_3}(z) + \text{Im} \int_{[x_n, w(t_n)]} \frac{f_3'}{f_3}(z) \\
&\leq O(1) + \text{Im} \int_{[x_n, w(t_n)]} \frac{f_3'}{f_3}(z)\,dz.
\end{aligned}$$

But if $z \in S(A)$ and $\text{Re}\,(z) = x_n$, and either $\text{Re}\,(z_k) > x_n + M$ or $\text{Re}\,(z_k) < -M$, then it is easy to check that

$$\text{Re}\left(\frac{1}{z - z_k} + \frac{1}{z_k}\right) < 0,$$

provided ϵ is sufficiently small (so that M is large). Since this accounts for all but at most finitely many of the zeros of f_3, we have by (3.11) that

$$(3.12)\qquad \text{Im} \int_{\sigma(0,t_n)} \frac{f_3'}{f_3}(z)\,dz \leq O(1) + \int_0^{\text{Im}\,(w(t_n))} \text{Re}\,\frac{f_3'}{f_3}(x_n + is)\,ds \leq O(1).$$

Combining (3.9), (3.10) and (3.12) gives that

$$\operatorname{Im} \int_{\sigma(0,t_n)} \frac{f'}{f}(z)\, dz \leq O(1) + (\epsilon O(1) - \delta C_0)x_n.$$

Taking ϵ to be sufficiently small compared to δ and letting $n \to \infty$ completes the proof.

We will use Lemma 3.2 to establish a fundamental property of the level sets (2.1) of a real entire function $f \in \mathfrak{S}(A)$ whose zeros also satisfy certain density conditions (cf. (3.14) and (3.15)). This property is that every level curve, γ, of f either has a branch point or "leaves" every horizontal strip $S(A)$, where $A > 0$.

THEOREM 3.3. *Let $f \in \mathfrak{S}(A_0)$. Suppose that there exists $\delta > 0$ such that*

$$\min\{d(f(z); x),\, d(f(-z); x)\} \geq \delta, \tag{3.14}$$

where $d(f; x)$ was defined in (1.5), and

$$\min\{Z(f; \{0 \leq Re(z) \leq x\}),\, Z(f; \{-x \leq Re(z) \leq 0\})\} \geq \delta x \tag{3.15}$$

for x sufficiently large. Fix any real number θ and let γ denote any level curve of f in the set (2.1) with $\gamma \subset \mathbb{H}^+$. Then either γ has a branch point or $\gamma \cap S(A)$ is a bounded subset of $\mathbb{H}^+$ for every $A > 0$.

Proof. Fix $A > 0$ and suppose γ has no branch points. Consider an unbounded simple curve σ such that $\sigma \subset \gamma$ and such that f has no zero on σ. It suffices to prove that $\sigma \cap S(A)$ is bounded. If not, then there is a parameterization of σ,

$$w : [t_0, \infty) \to S(A) \cap \mathbb{H}^+ \qquad (t_0 > 0)$$

for which we may assume without loss of generality that

$$f(w(t)) = e^{-i\theta}t, \qquad t \geq t_0. \tag{3.16}$$

(We know that $e^{i\theta}f$ is real and never 0 on σ. We have assumed in (3.16) that $e^{i\theta}f$ is positive; the following argument remains valid if $e^{i\theta}f$ is negative.) Also, by our assumptions, $\lim_{t\to\infty} |\mathrm{Re}\,(w(t))| = \infty$, and we may assume, by considering $f(-z)$ if necessary, that

$\lim_{t\to\infty} \operatorname{Re}(w(t)) = \infty$. Since (3.14) and (3.15) hold, we therefore see that the hypotheses of Lemma 3.2 are satisfied. Now, a calculation using (3.16) shows that for, $T > t_0$,

$$\operatorname{Im} \int\limits_{\sigma(t_0,T)} \frac{f'}{f}(z)\,dz = \operatorname{Im} \int_{t_0}^{T} \frac{e^{-i\theta}}{e^{-i\theta}t}\,dt = 0,$$

where, as before, $\sigma(t_0, T)$ is the portion of σ corresponding to $t_0 \leq s \leq T$. But this contradicts the conclusion of Lemma 3.2, and therefore σ cannot be unbounded and the proof is complete.

As a consequence of Theorem 3.3, we obtain the following corollary.

COROLLARY 3.4. *Suppose f satisfies the hypotheses of Theorem 3.3 and let θ be any real number. Let z_0 be a nonreal zero of f with $z_0 \in \mathbb{H}^+$ and let γ denote the level curve of f in the set (2.1) with $z_0 \in \gamma$. Then either γ has a branch point or there is a point $z_1 \in \gamma$ such that $\gamma \subset \{z \in \mathbb{C} \mid Im(z) \geq Im(z_1)\}$.*

Proof. Suppose γ is a level curve of f contained in the set (2.1) that passes through a zero $z_0 \in \mathbb{H}^+$ of f, and γ has no branch points. If θ is not an integral multiple of π, then since f is a real entire function, γ can only intersect the real axis at a zero of f. But, f can have at most one zero on γ, so γ cannot intersect the real axis. Thus, by Theorem 3.3, γ must leave every horizontal strip $S(A)$, $A > 0$. Hence, γ has a local minimum, z_1, as asserted. Finally, if θ is an integral multiple of π and γ does not intersect the real axis, then the same argument applies. On the other hand, if γ intersects the real axis, the point of intersection must be a branch point of γ since the real axis belongs to the set (2.1) when θ is an integral multiple of π. This contradicts the assumption that γ has no branch points, and completes the proof.

Remark. The assumption in Corollary 3.4 that there is a nonreal zero of f on γ is necessary. This can be seen by considering any $f \in \mathcal{L}-\mathcal{P}$ having only simple real zeros and satisfying (3.15). For then the components of the set of points where $\operatorname{Re}(f) = 0$ have no branch points or minima, as discussed in §2.

THEOREM 3.5. *Suppose f satisfies the hypotheses of Theorem 3.3, with zeros in the strip $S(A_0)$ for some $A_0 > 0$, and let θ be any real number. Then $f \in \mathcal{L}-\mathcal{P}$ if and only* *if* $L[f_{\mu,\theta}](x) > 0$ *for all* $x \in \mathbb{R}$ *and for all* μ *such that* $0 < \mu \leq A_0$.

Proof. The necessity of the condition for f to be in $\mathcal{L}-\mathcal{P}$ is just a restatement of Theorem 2.1 with $A = 0$. To prove the sufficiency, let $\theta \in \mathbb{R}$ and assume that f satisfies the hypotheses of Theorem 3.3, but that f is not in $\mathcal{L}-\mathcal{P}$. This means that f has a nonreal zero $z_0 \in \mathbb{H}^+$. By Corollary 3.4, the level curve γ of f in the set (2.1) with $z_0 \in \gamma$ either has a branch point or a horizontal tangent line at some point. In either case, by Lemma 2.2 there exist $x_1, \mu_1 \in \mathbb{R}$, $\mu_1 > 0$, such that $L[f_{\mu_1,\theta}](x_1) \leq 0$. By Theorem 2.1 (applied with $A = A_0$), $L[f_{\mu,\theta}](x) > 0$ if $\mu > A_0$. Thus $0 < \mu_1 \leq A_0$, and the proof is complete.

For the purpose of comparing this result to known theorems, we state two other theorems giving necessary and sufficient conditions for f to be in $\mathcal{L}-\mathcal{P}$.

THEOREM 3.6. ([6, Theorem 3.4]) *Let* f *be an even real entire function with zeros in* $S(A)$ *for some* $A > 0$ *and suppose that the Fourier transform of the restriction of* f *to* $\mathbb{R}$ *is positive and equal to* $O(\exp(-|x|^{2+\epsilon}))$ *for some* $\epsilon > 0$. *Then* $f \in \mathcal{L}-\mathcal{P}$ *if and only if* $L[f_{\mu,0}](x) + L[f_{\mu,\pi/2}](x) \geq 0$ *for all* $0 \leq \mu \leq A$ *and* $x \in \mathbb{R}$.

In contrast to Theorem 3.5, no density conditions on the number of real zeros of f are assumed in Theorem 3.6. On the other hand, the condition in Theorem 3.5 sufficient for f to be in $\mathcal{L}-\mathcal{P}$ is perhaps easier to verify, since it involves the Laguerre expression $f_{\mu,\theta}$ for any single choice of θ, rather than a sum of Laguerre expressions as in Theorem 3.6. We remark that the proof of Theorem 3.6 in [6] is based on the following theorem, due to Jensen, which involves a complex form of the Laguerre inequalities.

THEOREM 3.7. ([13], [P, p.17]) *Let* $f(z) = e^{-az^2}g(z)$, *where* g *is a real entire function of genus 0 or 1 and* $a \geq 0$. *Then* $f \in \mathcal{L}-\mathcal{P}$ *if and only if*

$$|f'(z)|^2 \geq Re\{f(z)\overline{f''(z)}\}, \quad z \in \mathbb{C}. \tag{3.17}$$

The proof of this theorem is based on the geometric interpretation of inequality (3.17), which is just that $|f(x+iy)|^2$ is a convex function of y.

4. Applications to the Riemann Hypothesis.

We conclude this paper with some applications to the Riemann ξ-function, which has a well-known Fourier integral representation. While the results of §3 may be applied to any real entire function which is the Fourier transform of a suitable kernel, for the sake of brevity, we will formulate our results only for the Riemann ξ-function.

To begin with, we will first review some of the notations and results which will be needed. If $\zeta(z)$ denotes the Riemann ζ-function, then it is well known [31, pp. 13, 30 and 45] that all the nonreal zeros of ζ lie in the critical strip $0 < \mathrm{Re}\,(z) < 1$ and are symmetrically located about the critical line $\mathrm{Re}\,(z) = 1/2$. Then the Riemann Hypothesis is the statement that all the nonreal zeros of ζ lie precisely on the critical line $\mathrm{Re}\,(z) = 1/2$. Now from Riemann's definition of his ξ-function, i.e.,

$$\xi(iz) = \frac{1}{2}(z^2 - \frac{1}{4})\pi^{\frac{z}{2}-\frac{1}{4}}\Gamma(\frac{z}{2} + \frac{1}{4})\zeta(z + \frac{1}{2}), \tag{4.1}$$

it can be seen that the Riemann Hypothesis is equivalent to the statement that all the zeros of ξ are real. It is also known that ξ admits the following integral representation (see [P, p. 11] or [31, p. 255])

$$H(x) = \frac{1}{8}\xi(\frac{x}{2}) = \int_0^\infty \Phi(t)\cos(xt)\,dt, \quad x \in \mathbb{C}, \tag{4.2}$$

where

$$\Phi(t) = \sum_{n=1}^{\infty} \pi n^2(2\pi n^2 e^{4t} - 3)\exp(5t - \pi n^2 e^{4t}), \quad t \in \mathbb{R}. \tag{4.3}$$

In addition, we recall that the even real entire function H is of order 1 and of maximal type (see [31, pp. 16 and 29] and [4, Appendix A]) and that all the zeros of H lie in the interior of $S(1)$ ([23], [2]). Thus $H \in \mathfrak{S}(1)$. We also note that Φ is an even function [23].

In order to apply the results of §3, we must verify that H satisfies the density conditions (3.14) and (3.15). This is easily deduced from the following classical results. First, the Riemann-von Mangoldt formula (see [7, pp. 18, 19, 301], [31, Chapter IX] or [12, Chapter 10]) asserts that

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T) \quad (T \to \infty), \tag{4.4}$$

where $N(T)$ denotes the number of zeros of $\zeta(z)$ in $0 < \mathrm{Re}\,(z) < 1$, $0 < \mathrm{Im}\,(z) \leq T$. Therefore, it follows from (4.1), (4.2) and (4.4) that H satisfies condition (3.15).

Now, let $N_0(T)$ denote the number of zeros of $\zeta(z)$ on the critical line, of the form $z = \frac{1}{2} + it$, $0 < t \leq T$. In 1942 A. Selberg [29] proved the existence of constants $T_0, C > 0$ such that

$$N_0(T) > CT \log T, \quad T \geq T_0.$$

Thus,

$$\kappa = \liminf_{T\to\infty} \frac{N_0(T)}{N(T)} > 0. \tag{4.5}$$

(For more recent results, see Levinson [17] ($\kappa > 1/3$) and Conrey [3] ($\kappa > 2/5$)). Thus it follows from (4.1), (4.2) and (4.5) that H satisfies condition (3.14). In conclusion, we see that H satisfies the hypotheses of Theorem 3.5. Therefore, the application of Theorem 3.5 allows us to deduce the following necessary and sufficient condition for the Riemann Hypothesis to be valid.

THEOREM 4.1. *Let $H(x)$ be defined by (4.2) and let θ be an arbitrary real number. Then $H \in \mathcal{L}-\mathcal{P}$ if and only if $L[H_{\mu,\theta}](x) > 0$ for all $x \geq 0$ and for all μ satisfying $0 < \mu < 1$.*

Proof. As before, the necessity of this condition for H to be in $\mathcal{L}-\mathcal{P}$ follows from Theorem 2.1 by taking A to be 0. As for the sufficiency, since the zeros of H are known to belong to $S(1)$, $H_{1,\theta} \in \mathcal{L}-\mathcal{P}$. Thus $L[H_{1,\theta}](x) \geq 0$ for all $x \in \mathbb{R}$, with equality only at a real zero of $H_{1,\theta}$ of multiplicity greater than one. But if x_0 is a multiple zero of $H_{1,\theta}$, then either $H'(x_0 + i) = 0$ or the component γ of the level set

$$\{z \in \mathbb{H}^+ \mid \operatorname{Re}(e^{i\theta} H(z)) = 0\}$$

that contains $x_0 + i$ has a horizontal tangent at this point (see the proof of Lemma 2.2). We will show that this leads to a contradiction.

We first show that

$$\operatorname{Im} \frac{H'}{H}(x_0 + i) = 0. \tag{4.6}$$

This is clear if $H'(x_0+i) = 0$, since H has no zero with imaginary part equal to 1. If γ has a horizontal tangent at $x_0 + i$, then we proceed, as in the proof of Lemma 2.3, by letting

$z(t)$ be a parameterization of γ with $z(t_0) = x_0 + i$. In this case, a short calculation (see (2.4)) shows that

$$\operatorname{Im} \frac{H'}{H}(x_0 + i) = \operatorname{Im} \frac{1}{t_0 z'(t_0)} = 0,$$

since the horizontal tangent of γ at $x_0 + i$ implies that $\operatorname{Im}(z'(t_0)) = 0$.

Since H is of genus 1, we have that H is the uniform limit on compact subsets of $\mathbb{C}$ of the functions p_n, where

$$p_n(z) = \prod_{k=1}^{n} (1 - z/z_k) e^{z/z_k}$$

and $\{z_k\}$ is the zero set of H. So $p_n \in \mathcal{L} - \mathcal{P}^*$, and (see (2.5))

$$\Lambda(p_n) = \{z \in \mathbb{H}^+ \mid \operatorname{Im} \frac{p_n'}{p_n}(z) \geq 0\} \subset \bigcup_{j=1}^{m} D(x_j, y_j),$$

where $z_j = x_j + iy_j$ are the nonreal zeros of p_n and as before $D(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}$. Since the zeros of H are in the interior of $S(1)$, we see that, for fixed $M > 0$, if $z_0 \in \Lambda(p_n)$ and $\operatorname{Re}(z_0) \leq M$, then $\operatorname{Im}(z_0) \leq 1 - \epsilon$, for some $\epsilon > 0$ that depends on M but not on n. By letting $n \to \infty$, we conclude that $\Lambda(H)$ is a subset of the interior of $S(1)$.

Thus (4.6) is the desired contradiction, and it follows that $H_{1,\theta}$ has only simple real zeros, and so $L[H_{1,\theta}](x) > 0$ for all $x \in \mathbb{R}$. Thus the hypothesis that $L[H_{\mu,\theta}](x) > 0$ for all $x \geq 0$ and for all μ satisfying $0 < \mu < 1$, together with the fact that H is an even function, implies that $L[H_{\mu,\theta}](x) > 0$ for all $x \in \mathbb{R}$ and all μ such that $0 < \mu \leq 1$. Theorem 3.5 now implies that $H \in \mathcal{L} - \mathcal{P}$, and the proof is complete.

Remark. The sufficient condition for the Riemann Hypothesis to hold in Theorem 4.1 only needs to be verified for x satisfying $|x| > 10^9$. This is an immediate consequence of the fact that all of the zeros of H in the set $\{z \mid |\operatorname{Re}(z)| \leq 1.0908\cdots \times 10^9\}$ are real [19] and the following proposition.

PROPOSITION 4.2. *Let $f \in \mathfrak{S}(A)$ and suppose that all zeros of f satisfying $|\operatorname{Re}(z)| \leq M$ are real. If $\mu \neq 0$ and $\theta \in \mathbb{R}$, then $L[f_{\mu,\theta}](x) > 0$ for all x such that $|x| \leq M - 2A$.*

Proof. Fix $\mu \neq 0$ and $\theta \in \mathbb{R}$. Suppose that $f \in \mathfrak{S}(A)$ and that all zeros of f satisfying $|\operatorname{Re}(z)| \leq M$ are real. Hence the union of the Jensen disks associated with the nonreal zeros of f is disjoint from the set $\{z \mid |\operatorname{Re}(z)| \leq M - A\}$. Since the nonreal zeros of f' are in these Jensen disks (cf. the proof of Theorem 4.1), it follows that all the zeros of f' in the set $\{z \mid |\mathrm{Re}\,(z)| \leq M - A\}$ are real.

Also, the nonreal zeros of $f_{\mu,\theta}$ are contained in the union the Jensen disks associated with f (see [2, Theorem 4]), and so the union of the Jensen disks associated with the nonreal zeros of $f_{\mu,\theta}$ is disjoint from the set $\{z \mid |\mathrm{Re}\,(z)| \leq M - 2A\}$.

A calculation similar to that carried out in (1.2) now shows that

$$L[f_{\mu,\theta}](x) = -f_{\mu,\theta}^2(x)\left(\frac{f'_{\mu,\theta}}{f_{\mu,\theta}}\right)'(x) < 0$$

only if x is on the boundary of the set

$$\Lambda = \{z \in \mathbb{H}^+ \mid \mathrm{Im}\,\frac{f'_{\mu,\theta}}{f_{\mu,\theta}}(z) > 0\}.$$

Since Λ is contained in the union of the Jensen disks associated with the nonreal zeros of $f_{\mu,\theta}$ (cf. the proof of Corollary 2.6), if $|x| \leq M - 2A$, then $L[f_{\mu,\theta}](x) \geq 0$. As before, equality is possible only at a multiple zero of $f_{\mu,\theta}$. As in the proof of Theorem 2.1, a multiple zero, x_0, of $f_{\mu,\theta}$ with $|x_0| \leq M - 2A$ corresponds to $x_0 + i\mu$ being a zero of f'. But we have already seen that all the zeros of $f'_{\mu,\theta}$ in the set $\{z \mid |\mathrm{Re}\,(z)| \leq M - 2A\}$ are real, so that $L[f_{\mu,\theta}](x) > 0$ if $|x| \leq M - 2A$, and the proof is complete.

COROLLARY 4.3. *Let $H(x)$ and $\Phi(t)$ denote the functions defined in (4.2) and (4.3), let θ be any real number and let*

$$F(x, \mu, \theta, s, t) = (\cos(2\theta)\cosh(2\mu t) + \cosh(2\mu s))\cos(xt) + \sin(2\theta)\sinh(2\mu t)\sin(xt).$$

The following are equivalent:

(i) $H \in \mathcal{L} - \mathcal{P}$;

(ii) *if* $0 < \mu < 1$ *and* $x > 10^9$, *then*

$$\int_0^\infty \int_0^\infty \Phi(s-t)\Phi(s+t)s^2 F(x, \mu, \theta, s, t)\, dsdt > 0\,;$$

(iii) *if* $0 < \mu < 1$ *and* $x > 10^9$, *then*

$$\int_0^\infty \int_0^\infty \Phi(s-t)\Phi(s+t)s^2[\cosh(2\mu t) + \cosh(2\mu s)]\cos(xt)\, dsdt > 0\,;$$

(iv) *if* $0 < \mu < 1$ *and* $x > 10^9$, *then*

$$\int_0^\infty \int_0^\infty \Phi(s-t)\Phi(s+t)s^2[\cosh(2\mu s) - \cosh(2\mu t)]\cos(xt)\, dsdt > 0\,.$$

Proof. An elementary, although somewhat lengthy, computation using (4.2) and the fact that Φ is an even function shows that

$$\frac{1}{8}L[H_{\mu,\theta}]\left(\frac{x}{2}\right)$$

is equal to the expression in (ii). Thus the equivalence of (i) and (ii) follows from Theorem 4.1 and the remark following its proof. The equivalences of (i) with (iii) and of (i) with (iv) follows by making the choices $\theta = 0$ and $\theta = \pi/2$ in (ii), respectively.

Observe that statements (iii) and (iv) above assert that certain Fourier transforms are positive. The theory of which kernels have positive Fourier transforms is well developed. Such kernels are called positive definite kernels. There are several known characterizations of nonnegative definite kernels. For example, the celebrated Tauberian theorem of Wiener (see [8] or [9]) asserts that a kernel K is nonnegative definite if and only if the closure of the span of translates of K is all of $L^1(\mathbb{R})$. For other characterizations see [21], [1], [18] and [14]. Thus the equivalence of statements (i), (iii) and (iv) of Corollary 4.3 may be restated as follows:

<u>COROLLARY 4.4.</u> *Let $H(x)$ and $\Phi(t)$ denote the functions defined in (4.2) and (4.3). The following are equivalent:*

(i) $H \in \mathcal{L}-\mathcal{P}$;

(ii) *if* $0 < \mu < 1$, *then*

$$\int_0^\infty \Phi(s-t)\Phi(s+t)s^2[\cosh(2\mu t)+\cosh(2\mu s)]\,ds$$

is a positive definite kernel;

(iii) *if* $0 < \mu < 1$, *then*

$$\int_0^\infty \Phi(s-t)\Phi(s+t)s^2[\cosh(2\mu s)-\cosh(2\mu t)]\,ds$$

is a positive definite kernel.

Remark. It may be of interest to compare the necessary and sufficient conditions for the Riemann Hypothesis to hold in Corollary 4.4 with the formulae of Jensen and Pólya [23, formulae (18), (19) and (20)].

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George Csordas
Wayne Smith
Department of Mathematics
University of Hawaii
Honolulu, HI 96822
USA

Richard S. Varga
Department of Mathematics and Computer Science
Kent State University
Kent, OH 44242
USA