An application from partial sums of $e^z$ to a problem in several complex variables

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Abstract


Let $B^n_r = \{ z \in \mathbb{C}^n : |z| < r \}$, where $| \cdot |$ is the Euclidean norm, and for $X \subset \mathbb{C}^n$, let $\mathcal{H}X$ denote the closed convex hull of $X$ in $\mathbb{C}^n$. In 1990, Graham showed that if $f$ is a normalized holomorphic map from $B^n_r$ into $\mathbb{C}^n$, and if $f$ is either an open map or a polynomial map, then there is a sharp, uniform constant $a$, $a$ given by $ae^1 + e = 1$, such that $\mathcal{H} f(B^n_r) \supset B^n_a$. Graham posed the question to find, for normalized polynomial maps $f$ of degree $m$, the best constant $a_m$ so that $\mathcal{H} f(B^n_r) \supset B^n_{a_m}$. We answer this question and obtain, for each $m$, the sharp constant

$$a_m = a + \frac{a \ln m}{2(1+a)m} + \frac{a \ln \left(\frac{\sqrt{2\pi} (1+a)}{a}\right)}{(1+a)m} + o\left(\frac{1}{m}\right), \quad m \to \infty.$$  

We also note that this solution extends an old result of Pólya and Szegő.

Keywords: Holomorphic maps in several complex variables; zeros of the partial sums of $e^z$; the Szegő curve.

In this short note, we show how a numerically motivated result on the zeros of the partial sums of $e^z$ in classical one complex variable function theory can be used to answer a question arising in several complex variables. To put the problem in context, we introduce the following

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notations. With $C^n := \{z = (z_1, z_2, \ldots, z_n) : z_k \in \mathbb{C} \text{ for } k = 1, 2, \ldots, n\}$, let $|z|^2 := \sum_{k=1}^{n} |z_k|^2$ and $B^n_r := \{z \in C^n : |z| < r\}$, with $B_r := B^1_r$, $B := B^1_1$ and $B^n := B^n_1$. We consider the classes
\[ \mathcal{S} := \{f : f \text{ is analytic and one-to-one on } B, \text{ with } f(0) = 0 \text{ and } f'(0) = 1\}, \]
\[ \mathcal{S}^* := \{f \in S : f(B) \text{ is starlike with respect to } z = 0\}, \]
\[ \mathcal{K} := \{f \in \mathcal{S} : f(B) \text{ is convex}\}. \]
Classical results (cf. [6]) obtained at the beginning of this century are as follows.

**Distortion theorem:**
\[ \frac{(1 - |z|)^{p-1}}{(1 + |z|)^{p+1}} \leq |f'(z)| \leq \frac{(1 + |z|)^{p-1}}{(1 - |z|)^{p+1}}, \quad z \in B; \quad (1) \]

**Growth theorem:**
\[ \frac{|z|}{(1 + |z|)^p} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^p}, \quad z \in B; \quad (2) \]

**Koebe Covering theorem:**
\[ f(B) \supset B_{1/2^p}, \quad (3) \]
where $p = 2$, for all $f$ in $\mathcal{S}$ and in $\mathcal{S}^*$. All three theorems are sharp when $p = 2$ for the function, known as the Koebe function, defined by
\[ f(z) := \frac{z}{(1 - z)^p}. \quad (4) \]
Corresponding results hold for functions $f \in \mathcal{K}$ with the exponent $p = 1$ in (1)--(4).

In 1933, in the appendix of Montel's book on Univalent Function Theory, Cartan [5] posed the question: Can the Distortion, Growth and Koebe Covering theorems be extended to one-to-one functions $f$ which are biholomorphic on $B^n$ and normalized by $f(0) = 0$ and $Jf(0) = I$, where $Jf$ denotes the Jacobian of $f$ and $I$ denotes the identity matrix? (The analogue of the Distortion theorem for $f$ in (1) would give bounds for the modulus of the determinant of the Jacobian $Jf$.) Cartan also explicitly asked if there were extensions to the convex and starlike subclasses of maps from $B^n$ into $\mathbb{C}^n$. In that appendix, he gave several examples of polynomial maps on $\mathbb{C}^n$. Since then, polynomial maps have been studied extensively in the literature of several complex variables.

That the three theorems cannot be directly extended to arbitrary biholomorphic maps from $B^n$ into $\mathbb{C}^n$ has been known for some time (cf. [7,9]). As a simple counterexample, consider the map $F(z_1, z_2) := (z_1, z_2 e^{\alpha z_1})$ for $\alpha \in \mathbb{R}$, so that $F$ is a normalized, one-to-one and biholomorphic mapping from $B^2$ into $\mathbb{C}^2$. It can be verified that for a suitable choice of $\alpha$, each of the four inequalities in (1) and (2), as well as the inclusion of (3), fails for this map $F$.

To answer Cartan's questions about the convex and starlike maps, the following extensions of the Distortion, Growth and Covering theorems were obtained in [2].
Theorem A. Let \( f \) be a normalized biholomorphic map from \( B^2 \) into \( \mathbb{C}^2 \) with \( f(B^2) \) a convex domain. Then, there exists a positive constant \( c \) such that
\[
\frac{(1 - |z|)^{-3/2}}{(1 + |z|)^{c+3/2}} \leq |\det Jf| \leq \frac{(1 + |z|)^{-3/2}}{(1 - |z|)^{c+3/2}}, \quad z \in B^2,
\]
where the constant \( c \) satisfies
\[
\frac{3}{2} \leq c < 1.71 \ldots . \tag{5}
\]

In [2], it was conjectured that \( c = \frac{3}{2} \). In [1], the next result was established.

Theorem B. Let \( f \) be a normalized biholomorphic map from \( B^n \) into \( \mathbb{C}^n \) with \( f(B^n) \) a starlike domain. Then,
\[
\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}, \quad z \in B^n,
\]
and
\[
f(B^n) \supseteq B_{1/4}^n.
\]
Furthermore, the above inequalities and inclusion are all sharp.

Although many of the classical results in \( \mathbb{C}^1 \) follow from each other, the situation in \( \mathbb{C}^n \) for \( n \geq 2 \) is different, where new techniques of proof must be used, and the results derived appear to be independent of each other.

Theorems A and B were also obtained and improved later in [8,13].

A further result, answering Cartan's question, was obtained by Graham in an interesting paper [9], where he proved the following.

Theorem C. Let \( f \) be a normalized holomorphic map from \( B^n \) into \( \mathbb{C}^n \) with either \( f \) being an open map or \( f \) being a polynomial map. Let \( a \) be the unique positive constant satisfying \( ae^{1+a} = 1 \), so that \( a = 0.278 \, 46 \ldots \). If \( \mathcal{H}X \) denotes the closed convex hull of \( X \), then \( \mathcal{H}f(B^n) \supseteq B_a^n \).

Note that if \( f(B^n) \) is convex, then \( f(B^n) \supset B_a^n \).

If we define the degree of a polynomial mapping \( \mathbb{C}^n \) into \( \mathbb{C}^n \) to be the maximal coordinate degree, then in [9], Graham posed the following question.

Question. If \( f \) is a polynomial map of degree \( m \), \( m \) a fixed integer, can an \( a = a_m \) be found such that
\[
\mathcal{H}f(B^n) \supseteq B_{a_m}^n ?
\]

In this paper, we link together sources from several fields and generate a solution for Graham's question.
A major tool used by Graham is a set of ideas introduced in [14]. It is shown in [14] that for each positive integer \( m \), there exists a positive constant \( a_m \) and a set \( \{ \alpha_j \}_{j=1}^{m} \), where \( \alpha_j \in \mathbb{C} \) for \( j = 1, 2, \ldots, m \), such that

(i) \[ |\alpha_j| \leq \frac{1}{a_m}, \]

(ii) \[ \sum_{j=1}^{m} \alpha_j = 1, \]

(iii) \[ \sum_{j=1}^{m} \alpha_j^k = 0, \quad k = 2, 3, \ldots, m, \quad \text{if } m > 1. \]

Together, (ii) and (iii) imply that, for any holomorphic \( f \) mapping \( B \) into \( \mathbb{C}^n \), the following representation formula is valid:

\[ \frac{1}{m} \sum_{j=1}^{m} f(\alpha_j z) = f(0) + zf'(0) + O(|z|^{m+1}), \]

for \( |z| < a_m \). The idea of the proof of [14] for the existence of \( \{ \alpha_j \}_{j=1}^{m} \) was to observe that the \( \alpha_j \)'s satisfying (ii) and (iii) are the zeros of the monic polynomials \( p_m \), defined by \( p_m(z) = z^m P_m(-1/z) \), where \( P_m(z) = \sum_{k=0}^{m} (mz)^k / k! \). If \( s_m(z) = \sum_{j=0}^{m} z^j / j! \) denotes the familiar partial sum of \( e^z \), then \( P_m(z) = s_m(mz) \). As shown in [1], if \( f \) is a polynomial of degree \( m \), then \( \mathcal{Z}(f(B)) \) contains the ball \( f(0) + zf'(0) \), \( |z| < a_m \). Thus, an answer can be given to Graham’s question by determining explicit bounds for the zeros of \( P_m(z) \).

In 1924, Szegő [12] defined a simple closed curve

\[ D_n := \{ z \in \mathbb{C} : |z e^{1-z}| = 1 \text{ and } |z| \leq 1 \}, \]

lying in the closed disk \( \overline{B} \). \( D_n \) has become known as the Szegő curve for the polynomials \( P_m \). If \( \{ z_{k,m}^{(m)} \}_{k=1}^{m} \) denotes the zeros of \( P_m \), it is known from the Eneström–Kakeya theorem (cf. [15, Chapter 4]) that \( \{ z_{k,m}^{(m)} \}_{k=1}^{m} \subset \overline{B} \) for each \( m \geq 1 \), and that \( \{ z_{k,m}^{(m)} \}_{k=1}^{m} \subset B \) for any \( m > 1 \). Clearly, the infinite set of all such zeros \( \{ z_{k,m}^{(m)} \}_{k=1}^{m} \) must possess at least one accumulation point in \( B \). In [12], Szegő showed that each accumulation point must lie on \( D_n \), and, conversely, that each point of \( D_n \) is an accumulation point of these zeros. Subsequently, it was shown in [3] that all of these zeros lie outside of \( D_n \). These facts are illustrated in Fig. 1.

Carpenter et al. [4] considered the problem of accurately estimating the zeros of \( P_m \). They introduced the arc defined by

\[ D_m := \{ z \in \mathbb{C} : |z e^{1-z}| = \frac{m! e^m}{m^m} \left| \frac{1-z}{z} \right|, \quad |z| \leq 1 \text{ and } \arg z \geq \cos^{-1}\left( \frac{m-2}{m} \right) \}, \]

for each \( m = 1, 2, \ldots \).

A careful examination of the arc \( D_m \) in [4] showed that, if \( \mathcal{C}_\delta := \{ z \in \mathbb{C} : |z - 1| < \delta \} \) for any \( \delta \) with \( 0 < \delta < 1 \), then the zeros \( \{ z_{k,m}^{(m)} \}_{k=1}^{m} \setminus \mathcal{C}_\delta \) are within \( O(1/m^2) \) of the arc \( D_m \). If it could have been shown that all the zeros \( \{ z_{k,m}^{(m)} \}_{k=1}^{m} \) of \( P_m \) lie outside \( D_m \) (which would have been the
natural analog of Buckholtz's result [3] for \( D_n \), then an answer to Graham's question would have followed. Indeed, numerical computations did suggest this, noting that in [4] the zeros of \( P_{27} \) appeared, up to plotting accuracy, to lie on the arc \( D_{27} \) (see Fig. 2). However, in an attempt to obtain more precise information about the zeros of \( P_m \) relative to \( D_m \), it was discovered that the zeros of \( P_m \) do not all lie outside \( D_m \) for every \( m \geq 1 \).

Indeed, it was shown in [16] that there exists a positive integer \( m_0 \), such that at least one zero of \( P_m \) does not lie outside \( D_m \) for every \( m > m_0 \). A direct calculation indicates the rather surprising outcome that \( m_0 = 96 \). The size of \( m_0 \) necessitated great precision in calculating the zeros. Brent’s MP package was used, with 120 significant digits, for these calculations.

Because of this result, it was natural to ask if a modification \( \hat{D}_m \) of the arc \( D_m \) could be found for which all of the zeros of \( P_m \) would be outside of \( \hat{D}_m \), for each \( m \geq 1 \). This was done recently in [16] where, for each \( m \geq 1 \), the arc \( \hat{D}_m \) was defined by

\[
\hat{D}_m = \left\{ z \in \mathbb{C} : |z e^{1-z}|^m = \frac{m! e^m}{m^m} \left| 1 - \text{Re} \frac{z}{z} \right|, |z| \leq 1 \text{ and } |\arg z| \geq \cos^{-1} \left( \frac{m-2}{m} \right) \right\},
\]

and it was shown that all the zeros of \( P_m \) do lie outside of \( \hat{D}_m \) for each \( m \geq 1 \) (see Fig. 3). This is the relevant result needed to provide an answer to Graham’s question. We include a brief outline of the proof of [16] that all the zeros of \( P_m \) lie outside of \( \hat{D}_m \) for each \( m \geq 1 \).
It is easy to verify by differentiation that

\[ e^{-z} s_m(z) = 1 - \frac{1}{m!} \int_0^z \xi^m e^{-\xi} \, d\xi, \]  

(11)

and, on replacing \( z \) by \( mz \) and \( \xi \) by \( m\xi \) and recalling that \( s_m(mz) = P_m(z) \), we have

\[ e^{-mz} P_m(z) = 1 - \frac{m^{m+1}}{m!} \int_0^z \xi^m e^{-m\xi} \, d\xi = 1 - \frac{m^{m+1}}{m!} e^{-m I_m(z)}, \]

(12)

where

\[ I_m(z) := \int_0^z (\xi e^{1-\xi})^m \, d\xi. \]

Using integration along the radial path \( \xi = \rho e^{i\theta}, 0 \leq \rho \leq r \), for the integral \( I_m(z) \), we obtain

\[ |I_m(z)| \leq \int_0^r (\rho e^{1-\rho \cos \theta})^m \, d\rho. \]

(13)

The integral on the right in (13) can be expressed as

\[ \frac{1}{|\cos \theta|^{m+1}} \int_0^r |\exp(1+r|\cos \theta|)| v^{m-1} \frac{\mu(v)}{(1 + \mu(v))} \, dv, \quad v := \mu e^{1+\mu}, \]
with careful use of the sign of \( \cos \theta \). Since \( \mu/(1 + \mu) \) is strictly increasing, it follows that

\[
|I_m(z)| < \frac{r(r e^{1-r \cos \theta})^m}{m(1 - r \cos \theta)} = \frac{|z| |ze^{1-z}|^m}{m(1 - \Re z)}, \quad 0 < r - |z| < 1, \ m = 1, 2, \ldots.
\]

Thus, if \( z_{k,m} \) is a zero of \( P_m \), then

\[
\frac{m^{m+1}}{m! e^m I_m(z_{k,m})} = 1
\]

from (12), so that (13) gives

\[
\frac{m^m |z_{k,m}| |z_{k,m} e^{1-z_{k,m}}|^m}{m! e^m (1 - \Re z_{k,m})} > 1.
\]

Therefore, all the zeros of \( P_m \) lie outside \( \hat{D}_m \) for all \( m \geq 1 \), as claimed.

We now determine the point of \( \hat{D}_m \), which is closest to the origin, i.e.,

\[
r_m := \min \{ |z| : z \in \hat{D}_m \}, \quad m = 1, 2, \ldots.
\]

Fixing any \( m \geq 1 \), let \( z = \rho(\phi) e^{i\phi} \) be any point of \( \hat{D}_m \). From (10),

\[
\left( \rho(\phi) e^{1 - \rho(\phi) \cos \phi} \right)^m = \frac{m! e^m (1 - \rho(\phi) \cos \phi)}{m^n \rho(\phi)}.
\]
since, by definition, \( \rho(\phi) \) satisfies \( \rho(\phi) \leq 1 \), which implies that \( 1 - \rho(\phi) \cos \phi \geq 0 \). On differentiating (15) as a function of \( \phi \), a straightforward calculation gives that

\[
\frac{d\rho(\phi)}{d\phi} = -\frac{\rho^2(\phi) \sin \phi \left[ m(1 - \rho(\phi) \cos \phi) - 1 \right]}{m(1 - \rho(\phi) \cos \phi)^2 + 1}. \tag{16}
\]

The definition in (10) implies that \( \cos \phi \leq (m - 2)/m \), showing that the quantity in brackets in (16) is at least unity for any \( m \geq 1 \). Thus, for \( z = \rho(\phi)e^{i\phi} \) on \( \hat{D}_m \) in the open upper half-plane, i.e., for \( \cos^{-1}((m - 2)/m) \leq \phi < \pi \), the derivative \( d\rho(\phi)/d\phi \) in (16) is negative. This establishes that the point of \( \hat{D}_m \), closest to the origin in the closed upper half-plane, is the intersection of the arc \( \hat{D}_m \) with the negative real axis, and, as the arc \( \hat{D}_m \) is, from (10), clearly symmetric about the real axis, the same is true for all of \( \hat{D}_m \).

Returning to Graham’s question, an \( a_m \), affirmatively answering Graham’s question, is defined by \( a_m := r_m \) where

\[
(r_m e^{1 + r_m})^m = \frac{m! \ e^m}{m^m} \left( \frac{1 + r_m}{r_m} \right), \quad m = 1, 2, \ldots. \tag{17}
\]

We remark that if \( a \) is the unique positive constant satisfying \( a e^{1+a} = 1 \), so that \( a = 0.27846 \ldots \), then it can be shown from (17) that

\[
r_m = a + \frac{a \ln m}{2(1 + a)m} + \frac{a \ln \left[ \sqrt{2\pi(1 + a)/a} \right]}{(1 + a)m} + o\left( \frac{1}{m} \right), \quad m \to \infty. \tag{18}
\]

We finally show that (18) is sharp in the following sense. It is well known that the partial sum \( s_n(z) \) of \( e^z \) has exactly one negative real zero if \( n \) is an odd positive integer, and that it has no real zeros if \( n \) is an even positive integer. If \( \{z_k(n)\}_{k=1}^\infty \) denotes the set of zeros of \( s_n(z) \) and if we write \( z_k(n) := r_k(n)e^{i\theta_k(n)} \) where \( |z_k(n)| = r_k(n) \) and where \( 0 < |\theta_k(n)| \leq \pi \), then from (11)

we have

\[
n! = e^{i(n+1)\theta_k(n)} \int_0^{r_k(n)} u^n e^{-u} \cos \theta_k(n) e^{-iu \sin \theta_k(n)} \, du, \quad n = 1, 2, \ldots. \tag{19}
\]

On taking moduli in (19),

\[
n! \leq \int_0^{r_k(n)} u^n e^{-u \cos \theta_k(n)} \, du, \quad n = 1, 2, \ldots,
\]

so that

\[
n! < \int_0^{r_k(n)} u^n e^u \, du, \quad \text{for any } \theta_k(n) \text{ with } 0 < |\theta_k(n)| < \pi. \tag{20}
\]

For \( n \) odd, say \( n := 2l + 1 \), let \( \tilde{r}_{2l+1} > 0 \) be such that \( s_{2l+1}(\tilde{r}_{2l+1}) = 0 \). From (19) and (20), we have

\[
(2l + 1)! \leq \int_0^{\tilde{r}_{2l+1}} u^{2l+1} e^u \, du < \int_0^{r_k(2l+1)} u^{2l+1} e^u \, du,
\]

which evidently implies that

\[
\tilde{r}_{2l+1} < r_k(2l + 1), \tag{21}
\]
for any zero \( z_k(2l + 1) \) with \(|\theta_k(2l + 1)| \neq \pi\). This shows that, for each odd positive integer \( m \), the unique negative real zero of \( P_m(z) = s_m(mz) \) is the closest zero of \( P_m(z) \) to the origin. In other words, the very best choice of \( a_m \), for every \( m = 2l + 1 \) odd, in Graham’s problem is just \( \tilde{r}_{2l+1} \). Now, in [11, II. Abschnitt, Exercise 215], it is shown that \( \tilde{r}_{2l+1} \) satisfies (18), with \( m \) replaced by \( 2l + 1 \). In this sense, the expression in (18), which holds for all integers \( m \), is sharp and extends, to the case of even integers, the result of [11].

References