Asymptotics for the zeros and poles of normalized Padé approximants to $e^z$

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Summary. With $s_n(z)$ denoting the $n$-th partial sum of $e^z$, the exact rate of convergence of the zeros of the normalized partial sums, $s_n(nz)$, to the Szegő curve $D_{0,\infty}$ was recently studied by Carpenter et al. (1991), where $D_{0,\infty}$ is defined by

$$D_{0,\infty} := \{ z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1 \}.$$ 

Here, the above results are generalized to the convergence of the zeros and poles of certain sequences of normalized Padé approximants $R_{n,\nu}(n+\nu)z$ to $e^z$, where $R_{n,\nu}(z)$ is the associated Padé rational approximation to $e^z$.

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1. Introduction

With $s_n(z) := \sum_{j=0}^{n} z^j/j!$, $n \geq 1$, denoting the familiar $n$-th partial sum of the exponential function $e^z$, it was shown in 1924 in a remarkable paper by Szegő [10] that the zeros \(\{z_n(k)\}_{k=1}^{n}\) of the normalized partial sum $s_n(nz)$, tend, as $n \to \infty$, to the closed curve $D_{0,\infty}$ in the closed unit disk, where

$$D_{0,\infty} := \{ z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1 \}.$$ 

Now, it is known (see [1] or [4]) that the zeros $\{z_n(k)\}_{k=1}^{n}$ all lie in the closed unit disk for every $n \geq 1$, and Szegő's result, more precisely, is that each accumulation point (in the closed unit disk) of all these zeros must lie on $D_{0,\infty}$, and, conversely, each point of $D_{0,\infty}$ is an accumulation point of these zeros!

Subsequently, the rate of convergence, as a function of $n$, of the zeros $\{z_n(k)\}_{k=1}^{n}$ to the curve $D_{0,\infty}$ was studied by Buckholtz [2] who showed, with the notation

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$$\text{dist } \left[ \left\{ z_n(k) \right\}_{k=1}^n; D_{0,\infty} \right] := \max_{1 \leq k \leq n} \left( \text{dist } \left[ z_n(k); D_{0,\infty} \right] \right),$$

that

$$\text{dist } \left[ \left\{ z_n(k) \right\}_{k=1}^n; D_{0,\infty} \right] \leq \frac{2e}{\sqrt{n}} \quad (n \geq 1),$$

which of course implies that

$$\lim_{n \to \infty} \left( \sqrt{n} \cdot \text{dist } \left[ \left\{ z_n(k) \right\}_{k=1}^n; D_{0,\infty} \right] \right) \leq 2e \approx 5.436 \ 563.$$

To complement the result of (1.3), it was later shown in Carpenter et al. [4] that

$$\lim_{n \to \infty} \left\{ \sqrt{n} \cdot \text{dist } \left[ \left\{ z_n(k) \right\}_{k=1}^n; D_{0,\infty} \right] \right\} = \text{Im } t_1 + \text{Re } t_1 \approx 0.636 \ 657,$$

where, denoting the complementary error function by

$$\text{erfc}(w) := \frac{1}{\sqrt{\pi}} \int_w^\infty e^{-t^2} \, dt \quad (w \in \mathbb{C}),$$

t_1 is the (complex) zero of \text{erfc}(w), in the upper half-plane, which is closest to the origin. From the numerical results of Fettis et al. [6], it is known that

$$t_1 \approx -1.354 \ 810 + i \ 1.991 \ 467.$$

Thus, if we express the upper bound of (1.2) as $O(1/\sqrt{n})$, as $n \to \infty$, then (1.4) shows that this upper bound is best possible in the sense that $1/\sqrt{n}$ cannot be replaced by a function of $n$ which tends more rapidly to zero, as $n \to \infty$, than does $1/\sqrt{n}$. (It is in this sense that we use the term best possible in what is to follow.)

It was also shown in [4] that a quantitatively faster convergence, of these zeros to $D_{0,\infty}$, takes place if one stays uniformly away from the point $z = 1$. Specifically, if we cover the point $z = 1$ with the open disk

$$C_\delta := \{ z \in \mathbb{C} : |z - 1| < \delta \} \quad (0 < \delta < 1),$$

then it was shown in [4, Theorem 2] that, for each fixed $\delta$ with $0 < \delta < 1$,

$$\text{dist } \left[ \left\{ z_n(k) \right\}_{k=1}^n \setminus C_\delta; D_{0,\infty} \right] = O \left( \frac{\ln n}{n} \right) \quad (n \to \infty),$$

where the constant, implicit in the right-side of (1.6), is dependent only on $\delta$.

For a more precise location of the zeros of $s_n(nz)$, consider the arc $D_{0,n}$, defined in [4] for each $n \geq 1$ by

$$D_{0,n} := \left\{ z \in \mathbb{C} : |ze^{1-z}|^n = \tau_n \sqrt{2\pi n} \frac{1-|z|}{\pi}, |z| \leq 1 \text{ and } |\arg z| \geq \cos^{-1} \left( \frac{n-1}{n} \right) \right\},$$

where $\tau_n$, from Stirling's formula, is given by the asymptotic series

$$\tau_n := \frac{n!}{\sqrt{2\pi n} n^ne^{-n}} \approx 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{5184n^3} + \cdots \quad (n \to \infty).$$

It was shown in [4, Proposition 3] that $D_{0,n}$ is a well-defined arc, and it is further shown in [4, Theorem 4] that, for each fixed $\delta$ with $0 < \delta < 1$,
(1.8) \[ \text{dist } \{ [z_n(k)]_{k=1}^n \setminus C_\delta : D_{0,n} \} = O \left( \frac{1}{n^2} \right) \quad (n \to \infty), \]

so that the arc \( D_{0,n} \) more closely approximates the zeros \( \{ z_n(k) \}_{k=1}^n \) of \( s_n(nz) \), than does the Szeg\'o curve \( D_{0,\infty} \). The results of (1.6) and (1.8) are both known to be best possible (cf. [4]).

Having reviewed the above results for the convergence behavior of the zeros of the normalized partial sums of \( e^z \), it is of course well known that the partial sums \( s_n(z) \) of \( e^z \) are just the special cases of the \((n,0)\)-th Pad\'e approximants to \( e^z \). The early result of Szeg\'o [10] has subsequently been generalized by Saff and Varga [9] to more general Pad\'e approximations, where one obtains, in the spirit of Szeg\'o, the convergence of the normalized zeros and poles of these Pad\'e approximants to the arcs \( D_{\sigma,\infty} \) and \( E_{\sigma,\infty} \), defined below in (1.16) and (1.17), in the closed unit disk. The goal of this paper is to obtain the analogs of (1.6) and (1.8) in this more general Pad\'e setting, thereby generalizing the results of [4] and [9].

In the remainder of this section, we introduce needed background and known results for this study of Pad\'e rational approximation to \( e^z \).

Let \( \pi_n \) denote the set of all complex polynomials of degree at most \( n \) \((n = 0, 1, \cdots)\). For each pair \((n, \nu)\) of nonnegative integers, the \((n, \nu)\)-th Pad\'e approximant to \( e^z \) is the rational function \( R_{n,\nu}(z) \) such that

\[ R_{n,\nu}(z) = \frac{P_{n,\nu}(z)}{Q_{n,\nu}(z)}, \]

where

(i) \( P_{n,\nu}(z) \in \pi_n \), and \( Q_{n,\nu}(z) \in \pi_\nu \) with \( Q_{n,\nu}(0) = 1 \), and

(ii) \( e^z - R_{n,\nu}(z) = O(|z|^{n+\nu+1}) \) as \( |z| \to 0 \).

It is well known (cf. Perron [8, p. 433], or Saff and Varga [9, p. 242]) that \( P_{n,\nu}(z) \) and \( Q_{n,\nu}(z) \) of (1.9) are given explicitly, for any pair \((n, \nu)\) of nonnegative integers \( n \) and \( \nu \), by

\[ P_{n,\nu}(z) = \sum_{k=0}^{n} \frac{(n+\nu-k)!n!z^k}{(n+\nu)!(n-k)!}, \]  

(1.10) and

\[ Q_{n,\nu}(z) = \sum_{k=0}^{\nu} \frac{(n+\nu-k)!\nu!(-z)^k}{(n+\nu)!(\nu-k)!}. \]  

(1.11)

The polynomials \( P_{n,\nu}(z) \) and \( Q_{n,\nu}(z) \) are respectively called the Pad\'e numerator and Pad\'e denominator of type \((n, \nu)\) for \( e^z \). In what follows, we consider, as in [9], any sequence of Pad\'e approximants \( \{ R_{n_j,\nu_j}(z) \}_{j=1}^\infty \) to \( e^z \) for which there exists a constant \( \sigma \), with \( 0 \leq \sigma < \infty \), such that

\[ \lim_{j \to \infty} n_j = \infty \quad \text{and} \quad \lim_{j \to \infty} \nu_j/n_j = \sigma. \]  

(1.12)

For any \( \sigma \) with \( 0 \leq \sigma < \infty \), define the two complex numbers

\[ z^\pm_{\sigma} := \left[ (1 - \sigma) \pm 2\sqrt{\sigma}i \right]/(1 + \sigma), \]  

(1.13)

which have modulus unity, and consider the complex plane \( \mathbb{C} \) slit along the two rays

\[ \mathcal{R}_{\sigma} := \{ z \in \mathbb{C} : z = z^+_\sigma + i\tau \quad \text{or} \quad z = z^-_\sigma - i\tau, \quad \text{for all } \tau \geq 0 \} , \]

\[ \mathcal{R}_{\sigma} := \{ z \in \mathbb{C} : \tau > 0 \} . \]  

as shown in Fig. 1.

With

\[ \lambda_\sigma := \frac{1 - \sigma}{1 + \sigma}, \]

the function \( g_\sigma(z) \), defined by

\[ g_\sigma(z) := \sqrt{1 + z^2 - 2\lambda_\sigma z}, \]  

has \( z_\sigma^2 \) as branch points, which are the finite extremities of \( \mathcal{R}_\sigma \). On setting \( g_\sigma(0) := 1 \) and on extending \( g_\sigma(z) \) analytically on the doubly slit domain \( \mathbb{C} \setminus \mathcal{R}_\sigma \), then \( g_\sigma(z) \) is analytic and single-valued on \( \mathbb{C} \setminus \mathcal{R}_\sigma \). It turns out that \( 1 \pm z + g_\sigma(z) \) does not vanish on \( \mathbb{C} \setminus \mathcal{R}_\sigma \) (cf. Saff and Varga [9, p. 244]).

Next, define \( (1 + z + g_\sigma(z))^{2/(1+\sigma)} \) and \( (1 - z + g_\sigma(z))^{2\sigma/(1+\sigma)} \) by requiring that their values at \( z = 0 \) be \( 2^{2/(1+\sigma)} \) and \( 2^{2\sigma/(1+\sigma)} \), respectively, and by analytic continuation. These functions are also analytic and single-valued on \( \mathbb{C} \setminus \mathcal{R}_\sigma \). For \( 0 < \sigma < \infty \), define the function \( w_\sigma(z) \) by

\[ w_\sigma(z) := \frac{4\sigma^{\sigma/(1+\sigma)} z e^{g_\sigma(z)}}{(1 + \sigma)(1 + z + g_\sigma(z))^{2/(1+\sigma)}(1 - z + g_\sigma(z))^{2\sigma/(1+\sigma)}} \quad (0 < \sigma < \infty). \]

Then, \( w_\sigma(z) \) is analytic and single-valued on \( \mathbb{C} \setminus \mathcal{R}_\sigma \), and is also univalent (and starlike) in \( |z| < 1 \) (cf. [9, p. 251]). We remark, on letting \( \sigma \to 0 \) in (1.15), that it is known (cf. [9, p. 244]) that \( w_0(z) := \lim_{\sigma \to 0} w_\sigma(z) \) satisfies

\[ w_0(z) = ze^{1-z} \quad (\text{for } |z| \leq 1), \]

which connects the above discussion to the Szegö curve of (1.1).

With the function \( w_\sigma(z) \) of (1.15) for \( 0 < \sigma < \infty \), the two Jordan arcs, \( D_{\sigma,\infty} \) and \( E_{\sigma,\infty} \), are defined by
(1.16) \( D_{\sigma, \infty} := \left\{ z \in \mathbb{C} : |w_\sigma(z)| = 1, \ |z| \leq 1, \ \text{and} \ \arg z \geq \cos^{-1} \left( \frac{1 - \sigma}{1 + \sigma} \right) \right\} \),

and

(1.17) \( E_{\sigma, \infty} := \left\{ z \in \mathbb{C} : |w_\sigma(z)| = 1, \ |z| \leq 1 \ \text{and} \ \arg z \leq \cos^{-1} \left( \frac{1 - \sigma}{1 + \sigma} \right) \right\} \),

where \(-\pi \leq \arg z \leq \pi\), and these arcs are symmetric with respect to the real axis. If

\[ J_{\sigma, \infty} := \{ z \in \mathbb{C} : |w_\sigma(z)| = 1 \ \text{and} \ |z| \leq 1 \}, \]

then we see from (1.16) and (1.17) that

\[ J_{\sigma, \infty} = D_{\sigma, \infty} \bigcup E_{\sigma, \infty} \quad (0 < \sigma < \infty). \]

Thus, for \( 0 < \sigma < \infty \), \( J_{\sigma, \infty} \) is a Jordan curve, consisting of the two Jordan arcs \( D_{\sigma, \infty} \) and \( E_{\sigma, \infty} \), and \( J_{\sigma, \infty} \) lies interior to the unit disk, except for the endpoints of these arcs, namely, \( z_\sigma^\pm \), which lie on the boundary of the unit disk. (For \( \sigma = 0 \), \( J_{0, \infty} \) reduces to the Szegő curve \( D_{0, \infty} \) of (1.1).)

With the arcs of (1.16) and (1.17), we have the following known result of [9]:

**Theorem A.** For any \( \sigma \) with \( 0 \leq \sigma < \infty \), consider any sequence of Padé approximants \( \{ R_{n_j, \nu_j}(z) \}_{j=1}^\infty \) to \( e^z \) for which (1.12) holds. Then,

(i) \( z \) is a limit point of the zeros of the normalized Padé approximants \( \{ R_{n_j, \nu_j}(n_j + \nu_j)z \}_{j=1}^\infty \) if and only if \( z \in D_{\sigma, \infty} \).

(ii) If \( 0 < \sigma < \infty \), then \( z \) is a limit point of the poles of the normalized Padé approximants \( \{ R_{n_j, \nu_j}(n_j + \nu_j)z \}_{j=1}^\infty \) if and only if \( z \in E_{\sigma, \infty} \).

The special case of (i) of Theorem A with \( \sigma = 0 \) and, in addition with \( n_j = j \) and \( \nu_j = 0 \) for all \( j \geq 1 \), reduces to Szegő’s result (cf. [9]). As previously mentioned, the convergence rates of the zeros in this case has been treated in detail in Carpenter et al. [4].

Since the polynomials of (1.10) and (1.11) satisfy the obvious identity

\[ Q_{n_j, \nu_j}(z) = P_{\nu_j, n_j}(-z), \]

it suffices then to investigate only the convergence behavior of the zeros of the normalized Padé approximants \( R_{n_j, \nu_j}(n_j + \nu) \), or equivalently, only the convergence behavior of the zeros of the normalized Padé numerators \( P_{n_j, \nu_j}(n_j + \nu) \). Clearly, all subsequent results for the zeros easily translate into results for the poles via (1.18).

### 2. Statements of new results

For any sequence of Padé approximants \( \{ R_{n_j, \nu_j}(z) \}_{j=1}^\infty \) to \( e^z \) which satisfies (1.12) with \( \sigma > 0 \), Theorem A above gives the precise location of the limit points of the zeros and poles of the normalized Padé approximants \( \{ R_{n_j, \nu_j}(n_j + \nu_j)z \}_{j=1}^\infty \). Our interest here is in determining the convergence behavior of zeros and poles of these normalized Padé approximants, as this would extend the results of [4] which are explicitly given for the case \( \sigma = 0 \). But, we note that the results of [4] were specifically...
determined for the special sequence \( \{ (n_j, \nu_j) \}_{j=1}^{\infty} \) with \( \nu_j := 0 \) and \( n_j := j \) for all \( j \geq 1 \), so that
\[
\nu_j / n_j = \sigma (= 0) \text{ for each } j \geq 1.
\]

On the other hand, when considering rational Padé approximants to \( e^z \) which are not polynomials, the second condition of (1.12) may hold, for the case \( \sigma > 0 \), for sequences \( \{ (n_j, \nu_j) \}_{j=1}^{\infty} \) having exceedingly slow convergence of \( \nu_j / n_j \) to \( \sigma \), as \( j \to \infty \). To indicate this, consider the sequence of pairs of positive integers \( \{ (n_j, \nu_j) \}_{j=3}^{\infty} \) defined by
\[
\nu_j := \left[ j \left( 1 + \frac{10}{\ln \ln j} \right) \right] \text{ and } n_j := j \text{ for all } j \geq 3,
\]
(where \([x]\) denotes the integer part of \( x \)), so that (1.12) is satisfied for \( \sigma = 1 \). But, for \( m := 10^6 \), we have
\[
n_m = 10^6 \text{ and } \nu_m = 4,808,374; \text{ whence, } \frac{\nu_m}{n_m} = 4.808 \, 374,
\]
which is far removed from the limiting value \( \sigma = 1 \). In this example, measuring the distance of the zeros or poles of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \), with respect to the limiting arc \( D_{1, \infty} \) or \( E_{1, \infty} \) of Theorem A, is of little value for \( j = 10^6 \).

Instead, we measure the distance of the zeros \( \{ z_{n_j, \nu_j}(k) \}_{k=1}^{n_j} \) of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \) from the Jordan arc \( D_{\sigma_j, \infty} \), where
\[
\sigma_j := \nu_j / n_j \text{ for all } j \geq 1,
\]
and where \( D_{\sigma_j, \infty} \) is the arc of (1.16) with \( \sigma \) replaced by \( \sigma_j \) of (2.1). Similarly, \( z_{\sigma_j}^\pm \) are defined from (1.13) with \( \sigma \) replaced by \( \sigma_j \), and, for a fixed \( \delta \) with \( 0 < \delta < 1 \), we set
\[
\delta_{\sigma_j} := \{ z \in \mathbb{C} : |z - z_{\sigma_j}^\pm| < \delta \} \cup \{ z \in \mathbb{C} : |z - z_{\sigma_j}^-| < \delta \},
\]
for all \( j \geq 1 \). We note that \( \delta_{\sigma_j} \), consisting of two disks, is the analog of \( C_\delta \) of (1.5) for the case \( \sigma = 0 \).

With the above definition, our first result (to be proved in Sect.3), which is patterned after the result of (1.6), can be stated as

**Theorem 1.** Consider any sequence of Padé approximants \( \{ R_{n_j, \nu_j}(z) \}_{j=1}^{\infty} \) to \( e^z \) for which
\[
\lim_{j \to \infty} n_j = \infty \quad \text{and} \quad \lim_{j \to \infty} \nu_j / n_j = \sigma, \quad \text{where } 0 < \sigma < \infty.
\]
If \( \{ z_{n_j, \nu_j}(k) \}_{k=1}^{n_j} \) denotes the zeros of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \), then for each fixed \( \delta \) with \( 0 < \delta < 1 \),
\[
\text{dist} \left( \{ z_{n_j, \nu_j}(k) \}_{k=1}^{n_j} \setminus \delta_{\sigma_j} ; D_{\sigma_j, \infty} \right) = O \left( \frac{1}{n_j + \nu_j} \right) \quad (j \to \infty).
\]
Moreover, the result of (2.4) is best possible.

We remark that the zeros of \( P_{n_j, \nu_j}((n_j + \nu_j)z) \) in (2.4), which are outside of the disks of \( \delta_{\sigma_j} \), are measured relative to the arc \( D_{\sigma_j, \infty} \), where both \( \delta_{\sigma_j} \) and \( D_{\sigma_j, \infty} \) in general vary with \( j \). Of course, there is a case where \( D_{\sigma_j, \infty} \) is a fixed arc for all \( j \geq 1 \), and this is covered in the following immediate corollary of Theorem 1.
Corollary 2. If, under the hypothesis of (2.3) of Theorem 1, \( \sigma \) is a positive rational number and if the associated sequence of nonnegative pairs of integers \( \{(n_j, \nu_j)\}_j^\infty \) satisfies
\[
\nu_j/n_j = \sigma \text{ for all } j \geq 1,
\]
then for each fixed \( \delta \) with \( 0 < \delta < 1 \),
\[
\text{dist } \{\{z_{n_j, \nu_j}(k)\}_{k=1}^{n_j}\}_{j=1}^\infty \setminus C_{\delta, \sigma}D_{\infty, \sigma} = O \left( \frac{1}{n_j + \nu_j} \right) \quad (j \to \infty).
\]

Moreover, the result of (2.6) is best possible.

We remark that the case \( \sigma = 0 \) of the zeros of the normalized partial sums of \( e^z \), as discussed in Sect. 1, are also measured, as in Corollary 2, against a fixed curve \( D_{0, \infty} \), but we note with interest that the result of (2.6) for \( 0 < \sigma < \infty \), which is the analog of (1.6), now eliminates the \( (\ln n) \) term appearing in (1.6). We also remark that essentially the special case \( \sigma = 1 \) of Corollary 2 is obtained (via a different technique) in [3].

To illustrate the results of Theorem 1 and Corollary 2, we have graphed in Fig. 2 the 8 zeros and 12 poles of \( R_{8,12}(20z) \), marked by \( \times \)'s, as well as the 16 zeros and 24 poles of \( R_{16,24}(40z) \), marked by dots, in relation to the arcs \( D_{3/2, \infty} \) and \( E_{3/2, \infty} \), for the case \( \sigma = 3/2 \). Note that the zeros and poles of \( R_{8,12}(20z) \) are, respectively, about twice as far from the curves \( D_{3/2, \infty} \) and \( E_{3/2, \infty} \), as are the zeros and poles of \( R_{16,24}(40z) \), which is in agreement with (2.4) of Theorem 1. Similar results are shown in Fig. 3, for \( R_{10,10}(20z) \) and \( R_{20,20}(40z) \) for the case \( \sigma = 1 \), and in Fig. 4, for \( R_{12,8}(20z) \) and \( R_{24,16}(40z) \) for the case \( \sigma = 2/3 \).

For our next result, we need some additional notation. As in [5, p. 22], for \( 0 < \sigma < \infty \) we set
\[
\hat{N}_\sigma(z) := \frac{g_\sigma(z) + 1 - \lambda_\sigma \cdot z}{z \sqrt{1 - \lambda_\sigma^2}} \quad (z \in \mathbb{C} \setminus (\mathbb{R}_\sigma \cup \{0\})�,
\]
where it is known that $\hat{N}_\sigma(z)$ is analytic and single-valued on $\mathbb{C} \setminus (\mathcal{R}_\sigma \cup \{0\})$. Then, in analogy with the arc $D_{0,n}$ of (1.7), we define, for each pair $(n_j, \nu_j)$ of positive integers, the arcs

$$D_{\sigma_j,n_j+\nu_j} := \{ z \in \mathbb{C} : |w_{\sigma_j}(z)|^{n_j+\nu_j} = |\hat{N}_{\sigma_j}(z)|, \; |z| \leq 1, \; \text{and} \; \arg z \geq \cos^{-1}\left(\frac{n_j - \nu_j - 2}{n_j + \nu_j}\right) \} \tag{2.8}$$

and

$$E_{\sigma_j,n_j+\nu_j} := \{ z \in \mathbb{C} : |w_{\sigma_j}(z)|^{n_j+\nu_j} = |\hat{N}_{\sigma_j}(z)|, \; |z| \leq 1, \; \text{and} \; \arg z \leq \cos^{-1}\left(\frac{n_j - \nu_j + 2}{n_j + \nu_j}\right) \} \tag{2.9}$$

where $\sigma_j := \nu_j/n_j$ and where $-\pi \leq \arg z < +\pi$. It is shown in Sect. 4 that these arcs of (2.8) and (2.9) are well-defined.

With the above definitions, our next result (to be proved in Sect. 5), which is patterned after the result of (1.8), can be stated as

**Theorem 3.** Under the hypothesis of (2.3) of Theorem 1,

$$\text{dist} \left[ \left\{ z_{n_j,\nu_j}(k) \right\}_{k=1}^{n_j} \setminus \mathcal{C}_{\sigma,\sigma_j} \right] \cap D_{\sigma_j,n_j+\nu_j} = O\left(\frac{1}{(n_j + \nu_j)^2}\right) \quad (j \to \infty). \tag{2.10}$$

We remark that a special case of (2.10) of Theorem 3 was previously established in de Bruin et al. [5]. Specifically, for the case $\sigma = 1$ and $n_j$ odd for all $j \geq 1$ of (2.3), it was shown in [5, eq. (9.31)] that the negative real zero $z_{n_j,\nu_j}$ of $R_{n_j,\nu_j}(n_j + \nu_j)$ satisfies

$$z_{n_j,\nu_j} = \hat{z}_{n_j,\nu_j} + O\left(\frac{1}{(n_j + \nu_j)^2}\right) \quad (j \to \infty) \tag{2.11}$$

where $\hat{z}_{n_j,\nu_j}$ denotes the real point of the arc $D_{\sigma_j,n_j+\nu_j}$. We also remark that the result of Theorem 3, for essentially the case $\sigma_j = 1$, is obtained in [3].
Fig. 4. Zeros and poles of $R_{12,8}(20z)$ and $R_{24,16}(40z)$, and the arcs $D_{2/3,\infty}$ and $E_{2/3,\infty}$.

Fig. 5. Zeros and poles of $R_{16,24}(40z)$, and the arcs $D_{3/2,40}$ and $E_{3/2,40}$. 

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Fig. 6. Zeros and poles of $R_{15,15}(30\omega)$, and the arcs $D_{1,30}$ and $E_{1,30}$

Fig. 7. Zeros and poles of $R_{25,10}(40\omega)$, and the arcs $D_{3/3,40}$ and $E_{2/3,40}$
To illustrate the result of Theorem 3, we have graphed in Figs. 5–7 the following cases. For the normalized Padé approximant \( R_{16,24}(40z) \) to \( e^z \), for which \( n_j = 16 \) and \( \nu_j = 24 \) and \( \sigma_j = 3/2 \), we show in Fig. 5 the curves \( D_{3/2,40} \) and \( E_{3/2,40} \), along with the 16 zeros and 24 poles (marked by the *'s) of \( R_{16,24}(40z) \). Figs. 6 and 7 show the corresponding results for \( R_{15,15}(30z) \) and \( R_{24,16}(40z) \). Up to plotting accuracy, it appears that the zeros and poles of these normalized Padé approximants lie on the respective arcs \( D_{\sigma_j, n_j + \nu_j} \) and \( E_{\sigma_j, n_j + \nu_j} \).

3. Proof of Theorem 1

We begin with the following

**Lemma 1.** For any \( \tau \) with \( 0 < \tau < \infty \), consider the sectorial set

\[
S_\tau := \left\{ z = re^{i\psi} : 0 < \tau \leq 1 \text{ and } \cos^{-1}\left(\frac{1 - \tau}{1 + \tau}\right) \leq \psi \leq 2\pi - \cos^{-1}\left(\frac{1 - \tau}{1 + \tau}\right) \right\},
\]

(3.1)

as shown in Fig. 8. Then,

\[
\min\{|\tilde{N}_\tau(z)| : z \in S_\tau\} = 1,
\]

(3.2)

with equality holding only at the points \( z^{\pm}_\tau \). Thus (cf. (2.2)), for any fixed \( \delta \) with \( 0 < \delta < 1 \),

\[
\min\{|\tilde{N}_\tau(z)| : z \in S_\tau \setminus \hat{C}_{\delta, \tau}\} > 1.
\]

(3.3)

**Proof.** From the definitions of (2.7) and (1.13) and the fact that \( g_\tau(z^{\pm}_\tau) = 0 \), it is readily verified that

\[
|\tilde{N}_\tau(z^{\pm}_\tau)| = 1.
\]

(3.4)

Next, as mentioned in de Bruin et al. [5, p. 22], \(|\tilde{N}_\tau(e^{i\psi})|\) is strictly increasing in \( \psi \) on the interval \([\cos^{-1}\left(\frac{1 - \tau}{1 + \tau}\right), \pi]\) and \(|\tilde{N}_\tau(e^{i\psi})|\) is strictly decreasing in \( \psi \) on the interval \([\pi, 2\pi - \cos^{-1}\left(\frac{1 - \tau}{1 + \tau}\right)]\). In addition (cf. [5, p.22]), for any fixed \( \psi \) with \( \cos^{-1}\left(\frac{1 - \tau}{1 + \tau}\right) \leq \psi \leq 2\pi - \cos^{-1}\left(\frac{1 - \tau}{1 + \tau}\right) \), \(|\tilde{N}_\tau(re^{i\psi})|\) is strictly decreasing in \( r \) on the interval \( 0 < r \leq 1 \), where we note from (2.7) that \( |\tilde{N}_\tau(0)| = +\infty \). With (3.4) and the definition of (3.1), we see geometrically that (3.2) is valid, where from (3.4), equality holds in (3.2) only at the points \( z^{\pm}_\tau \). Finally, since \( \hat{C}_{\delta, \tau} \), from its definition in (2.2), contains the points \( z^{\pm}_\tau \), then (3.3) follows directly from (3.2). \( \Box \)

This brings us to the

**Proof of Theorem 1.** With the hypothesis of (2.3) of Theorem 1, it is known (cf. [5, eq. (9.24)]) and [9, eq. (4.30)]) that if \( z \) is any zero of \( R_{n_j, \nu_j} ((n_j + \nu_j)z) \), then

\[
|w_{\sigma_j}(z)|^{n_j+\nu_j} = |\tilde{N}_{\sigma_j}(z)| \cdot \left\{ 1 + O\left(\frac{1}{n_j + \nu_j}\right) \right\} \quad (j \to \infty),
\]

(3.5)

uniformly on any compact subset of \( \mathbb{C} \setminus (\mathcal{R}_\sigma \cup \{0\}) \). Furthermore, it is known (cf. [9, Theorem 1.1]) that, for \( \nu_j \geq 0 \) and \( n_j \geq 2 \), all zeros of \( R_{n_j, \nu_j} ((n_j + \nu_j)z) \) must lie in the infinite sector

\[
\left\{ z \in \mathbb{C} : |\arg z| > \cos^{-1}\left(\frac{n_j - \nu_j - 2}{n_j + \nu_j}\right) \right\},
\]
and that (cf. [9, Theorem 2.3]) these zeros must have a positive density in the sectorial set $S_{\sigma}$.

For the zero $z$ of $R_{j\zeta}((n_j + \nu_j)z)$, we may assume from the above discussion that $z$ is in $S_{\sigma}$, and that $|z| < 1$. Writing $z = re^{i\theta}$, let $\bar{z} := \bar{r}e^{i\theta}$ be such that $|w_{\sigma_j}(\bar{z})| = 1$, i.e., $\bar{z} \in D_{\sigma_j, \infty}$. On defining $z - \bar{z} = se^{i\theta}$, |s| then measures the distance between $z$ and $\bar{z}$. Because $w_{\sigma_j}(z)$ is analytic and single-valued on $\mathbb{C} \setminus \mathcal{R}_{\sigma_j}$, the Taylor expansion of $w_{\sigma_j}(z)$ about the point $\bar{z}$ is

$$w_{\sigma_j}(z) = w_{\sigma_j}(\bar{z}) + se^{i\theta}w'_{\sigma_j}(\bar{z}) + O(s^2),$$

and, since the nearest singularities of $w_{\sigma_j}(z)$ to $\bar{z}$ in $\mathbb{C} \setminus \mathcal{R}_{\sigma_j}$ occur at the points $z_{\sigma_j}^\pm$, this Taylor expansion is convergent in the open disk with center $\bar{z}$ and radius

$$\min\{|\bar{z} - z_{\sigma_j}^+, |\bar{z} - z_{\sigma_j}^-|\}.$$

With [9, eq. (4.2)], it is known in general that

$$w'(u) = w_\tau(u)g_\tau(u)/u \quad \text{(any } 0 \leq \tau < \infty, \text{ any } u \in \mathbb{C} \setminus \mathcal{R}_{\tau}).$$

Thus, with $\tau = \sigma_j$ and $u = \bar{z}$ in (3.7), we can express (3.6), on factoring out $w_{\sigma_j}(\bar{z})$, as

$$w_{\sigma_j}(z) = w_{\sigma_j}(\bar{z}) \left\{ 1 + \frac{8g_{\sigma_j}(\bar{z})}{\bar{r}} + O(s^2) \right\},$$

where, since $w_{\sigma_j}(0) = 0$ from (1.15), this modified Taylor expansion in (3.8) is now convergent in the open disk with center $\bar{z}$ and radius

$$\min\{|\bar{z} - z_{\sigma_j}^+, |\bar{z} - z_{\sigma_j}^-|; |\bar{z}|\}.$$

Similarly, we expand $\hat{N}_{\sigma_j}(z)$ in a Taylor series about $\bar{z}$. Because in general (cf. [5, eq.(9.23)]),
Asymptotics for the zeros and poles of normalized Padé approximants to $e^s$

\begin{equation}
(3.10) \quad \hat{N}'(u) = -\hat{N}(u)/(g(\tau) \cdot u) \quad \text{any } 0 < \tau < \infty, \text{ any } u \in \mathbb{C}\setminus(\mathbb{R}_\tau \cup \{0\}),
\end{equation}

the Taylor expansion for $\hat{N}_{\sigma_j}(z)$ can be analogously expressed as

\begin{equation}
(3.11) \quad \hat{N}_{\sigma_j}(z) = \hat{N}_{\sigma_j}(\bar{z}) \left\{ 1 - \frac{s}{\hat{\tau} g_{\sigma_j}(\bar{z})} + O(s^2) \right\},
\end{equation}

where, as $\hat{N}_{\sigma_j}(z)$ is analytic and single-valued in $\mathbb{C}\setminus(\mathbb{R}_\sigma_j \cup \{0\})$, the modified Taylor expansion of (3.11) is convergent in the same open disk with center $\bar{z}$ and radius given by (3.9). With the expressions of (3.8) and (3.11), we derive from (3.5), since $|w_{\sigma_j}(\bar{z})| = 1$, that

\[
\left\{ 1 + \frac{s}{\hat{\tau}} \text{Re} \left( g_{\sigma_j}(\bar{z}) \right) + O(s^2) \right\}^{n_j + \nu_j} = |\hat{N}_{\sigma_j}(\bar{z})| \left\{ 1 - \frac{s}{\hat{\tau}} \text{Re} \left( \frac{1}{g_{\sigma_j}(\bar{z})} \right) + O(s^2) \right\} \left\{ 1 + O \left( \frac{1}{n_j + \nu_j} \right) \right\}.
\]

On taking logarithms in the above display and on dividing by $(n_j + \nu_j)$, we obtain

\[
\ln \left\{ 1 + \frac{s}{\hat{\tau}} \text{Re} \left( g_{\sigma_j}(\bar{z}) \right) + O(s^2) \right\} = \frac{\ln |\hat{N}_{\sigma_j}(\bar{z})|}{(n_j + \nu_j)} + \frac{1}{(n_j + \nu_j)} \ln \left\{ 1 - \frac{s}{\hat{\tau}} \text{Re} \left( \frac{1}{g_{\sigma_j}(\bar{z})} \right) + O(s^2) \right\} + O \left( \frac{1}{(n_j + \nu_j)^2} \right),
\]

and for $s$ small, this reduces to

\[
\frac{s}{\hat{\tau}} \text{Re} \left( g_{\sigma_j}(\bar{z}) \right) + O(s^2) = \frac{\ln |\hat{N}_{\sigma_j}(\bar{z})|}{(n_j + \nu_j)} - \frac{s \text{Re}(1/g_{\sigma_j}(\bar{z}))}{(n_j + \nu_j)} + O \left( \frac{1}{(n_j + \nu_j)^2} \right).
\]

Thus, we see that

\begin{equation}
(3.12) \quad s = \frac{(\ln |\hat{N}_{\sigma_j}(\bar{z})|)\hat{\tau}}{(n_j + \nu_j) \cdot \text{Re} \left( g_{\sigma_j}(\bar{z}) \right)} + O \left( \frac{1}{(n_j + \nu_j)^2} \right) \quad (j \to \infty).
\end{equation}

Now from [9, eq. (4.1)], it is known that $\text{Re}(g(\tau)) > 0$ on $\mathbb{C}\setminus\mathbb{R}_\tau$, and as $g(\tau)$ vanishes only at its branch points $z^\pm = \frac{\pm 1}{\tau}$, then $1/\text{Re}(g(\tau))$ is uniformly bounded at all points of the unit disk not in $\mathbb{C}_{\delta,\tau}$. Next, since $|N(\tau)| > 1$ on $S_\delta \setminus \mathbb{C}_{\delta,\tau}$ from Lemma 1 and since (cf. (1.16)) $D_{\tau,1}$ never passes through $z = 0$ because $w_{\tau}(0) = 0$, it follows from (3.12) that

\begin{equation}
(3.13) \quad s = O \left( \frac{1}{n_j + \nu_j} \right) \quad \text{for any zero } z \text{ of } R_{n_j + \nu_j}((n_j + \nu_j)z) \text{ not in } \mathbb{C}_{\delta,\sigma_j}
\end{equation}

($j \to \infty$). But as $|s|$ measures the distance from $z$ to a particular point, $\bar{z}$, of $D_{\sigma_j,1}$, then $\text{dist} \left\{ z; D_{\sigma_j,1} \right\} \leq |s|$ for any zero $z$ of $R_{n_j + \nu_j}((n_j + \nu_j)z)$ not in $\mathbb{C}_{\delta,\sigma_j}$ and it follows from (3.13) that

\[
\text{dist} \left\{ [z_{n_j + \nu_j}(\bar{z})]_{k=1}^{n_j + \nu_j} \setminus \hat{C}_{\delta,\sigma_j}; D_{\sigma_j,1} \right\} = O \left( \frac{1}{n_j + \nu_j} \right) \quad (j \to \infty),
\]
which is the desired result of (2.4) of Theorem 1.

We also remark that since all the factors, appearing in the first term on the right in (3.12), are positive, then \( s > 0 \) (for all \( j \) sufficiently large), which means, from our construction, that the associated zeros of \( R_{\nu_j}(n_j + \nu_j)z \) must lie to the left of the arc \( D_{\sigma_j, \infty} \). Similarly, because of (1.18), the associated poles of \( R_{\nu_j}(n_j + \nu_j)z \) must lie to the right of the arc \( E_{\sigma_j, \infty} \). This can be explicitly seen in Figs. 2-4.

Finally, to show that the result of (2.4) of Theorem 1 is sharp, the multiplier of \((n_j + \nu_j)^{-1}\) in the first term on the right in (3.12) is but a special case of \( \sigma_j = \tau \) of

\[
\frac{(\ln |\tilde{N}_{\tau}(\tilde{z})|)}{\text{Re} (g_{\tau}(\tilde{z}))}, \quad \text{where} \quad \tilde{z} \in D_{\tau, \infty} \setminus \tilde{C}_{\delta, \tau},
\]

But from the discussion above, it also follows that, for any \( \tau \) with \( 0 < \tau < \infty \) and any fixed \( \delta \) with \( 0 < \delta < 1 \), there exist constants \( M_1(\tau) \) and \( M_2(\tau, \delta) \) such that

\[
(3.14) \quad 0 < M_1(\tau) \leq \frac{(\ln |\tilde{N}_{\tau}(\tilde{z})|)}{\text{Re} (g_{\tau}(\tilde{z}))} \leq M_2(\tau, \delta) \quad \text{for all} \quad \tilde{z} \in D_{\tau, \infty} \setminus \tilde{C}_{\delta, \tau}.
\]

Hence, because \( \sigma_j \to \sigma \) as \( j \to \infty \) (where \( 0 < \sigma < \infty \)) and because of the bounds of (3.14), it follows that the first term on the right in (3.12) is exactly of order \((n_j + \nu_j)^{-1}\), as \( j \to \infty \), which shows that the result (2.4) of Theorem 1 is sharp. \( \square \)

4. The arcs \( D_{\sigma_j, n_j + \nu_j} \) and \( E_{\sigma_j, n_j + \nu_j} \)

Here, we show that the arcs \( D_{\sigma_j, n_j + \nu_j} \) and \( E_{\sigma_j, n_j + \nu_j} \), defined in (2.8) and (2.9), are well-defined for \( j \) sufficiently large, where we assume, as in Theorem 1, that (2.3) is valid. Because the treatment of the arcs \( E_{\sigma_j, n_j + \nu_j} \) is similar, we consider below only the arcs \( D_{\sigma_j, n_j + \nu_j} \).

For a given \( \tau \) with \( 0 < \tau < \infty \), consider the function defined by

\[
U_{\tau, m}(z) := (w_{\tau}(z))^m / \tilde{N}_{\tau}(z)
\]

for any positive integer \( m \). It can be verified that \( U_{\tau, m}(z) \) is analytic and single-valued on \( \mathbb{C} \setminus \mathcal{R}_{\tau} \). On fixing any \( \theta \) with

\[
\cos^{-1} \left( \frac{1 - \tau}{1 + \tau} \right) < \theta < 2\pi - \cos^{-1} \left( \frac{1 - \tau}{1 + \tau} \right),
\]

we know from the discussion in Sect. 3 that \( |U_{\tau, m}(re^{i\theta})| \) is strictly increasing on the interval \( 0 \leq r \leq 1 \), where \( U_{\tau, m}(0) = 0 \). In addition, since the arc \( D_{\tau, \infty} \) lies completely in the open unit disk (except for its endpoints), we have that \( |w_{\tau}(e^{i\theta})| > 1 \), and from Lemma 1, we similarly have \( |\tilde{N}_{\tau}(e^{i\theta})| > 1 \). It follows from (4.1) that, for all \( m \) sufficiently large, say \( m \geq m_0(\tau, \theta) \),

\[
|U_{\tau, m}(e^{i\theta})| > 1 \quad (m \geq m_0(\tau, \theta)).
\]

But the strict increase of \( |U_{\tau, m}(re^{i\theta})| \), as a function of \( r \) on \([0, 1]\), gives that there is a unique \( \hat{r} = \hat{r}(\theta, \tau, m) \) with \( 0 < \hat{r} < 1 \), such that

\[
|U_{\tau, m}(\hat{r}e^{i\theta})| = 1,
\]
i.e., from (2.8), \( r e^{i\theta} \) necessarily lies on the arc \( D_{\sigma, m} \). It is also evident that \( r(\theta, \tau, m') \prec r(\theta, \tau, m) \) for \( m' > m \geq m_0(\tau, \theta) \), since, by definition, \( |w_\tau(r e^{i\theta})|^m = |\hat{N}_\tau(r e^{i\theta})| > 1 \), where the last inequality follows from Lemma 1.

Next, with the assumption of (2.3) in Theorem 1, we see that the arcs \( D_{\sigma, \infty} \), defined in (1.16), converge, uniformly as \( j \to \infty \), to the fixed arc \( D_{\sigma, \infty, 0} \), where \( 0 < \sigma < \infty \). But as the arcs \( D_{\sigma, \infty} \) lie in the open unit disk (with the exception of its endpoints) for any \( j \), then for any \( \epsilon > 0 \) sufficiently small and for any \( \theta \) satisfying \( \cos^{-1}(\frac{1-\sigma}{1+\sigma}) + \epsilon \leq \theta \leq 2\pi - \cos^{-1}(\frac{1-\sigma}{1+\sigma}) - \epsilon \), it follows that \( |w_{\sigma j}(e^{i\theta})| > 1 \) for all \( j \) sufficiently large, so that

\[
|w_{\sigma j}(e^{i\theta})|^{n_j + \nu_j} / |\hat{N}_{\sigma j}(e^{i\theta})| > 1 \quad \text{for all } j \text{ sufficiently large.}
\]

Hence, the argument above shows that, for all \( j \) sufficiently large, there is an \( r(j) \) with \( 0 < r(j) < 1 \) such that \( |w_{\sigma j}(r(j)e^{i\theta})|^{n_j + \nu_j} = |\hat{N}_{\sigma j}(r(j)e^{i\theta})| \), i.e., \( r(j)e^{i\theta} \) lies on \( D_{\sigma j, n_j + \nu_j} \). This thus establishes

**Lemma 2.** Under the hypothesis of (2.3) of Theorem 1, the arcs \( D_{\sigma j, n_j + \nu_j} \) and \( E_{\sigma j, n_j + \nu_j} \) of (2.8) and (2.9), are well-defined, for all \( j \) sufficiently large.

### 5. Proof of Theorem 3

Under the hypothesis of (2.3) of Theorem 1, the sets \( D_{\sigma j, n_j + \nu_j} \) and \( E_{\sigma j, n_j + \nu_j} \) are well-defined from Lemma 2 for all \( j \) sufficiently large, say \( j \geq j_0 \). For \( j \geq j_0 \), let \( z \) be any zero of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \) in the unit disk and write \( z = re^{i\theta} \). Then, let \( \hat{z} = \hat{r}e^{i\theta} \) be on the arc \( D_{\sigma j, n_j + \nu_j} \) and, as before, set \( z - \hat{z} = se^{i\theta} \), where \(|s| \) measures the distance between \( z \) and \( \hat{z} \). Since \( z \) is a zero of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \), we have from (3.5) that

\[
|w_{\sigma j}(z)|^{n_j + \nu_j} = |\hat{N}_{\sigma j}(z)| \cdot \left\{ 1 + O\left(\frac{1}{n_j + \nu_j}\right) \right\} \quad (j \to \infty),
\]

or, in the notation of (4.1),

\[
|U_{\sigma j, n_j + \nu_j}(z)| = 1 + O\left(\frac{1}{n_j + \nu_j}\right), \quad (j \to \infty).
\]

Similarly, as in the proof of Theorem 1 in Sect. 3, we expand \( U_{\sigma j, n_j + \nu_j}(z) \) in a Taylor series about \( \hat{z} \), i.e.,

\[
U_{\sigma j, n_j + \nu_j}(z) = U_{\sigma j, n_j + \nu_j}(\hat{z}) + se^{i\theta}U'_{\sigma j, n_j + \nu_j}(\hat{z}) + O(s^2).
\]

Using the definition of \( U_{\sigma j, n_j + \nu_j}(\hat{z}) \) in (4.1), along with the identities of (3.7) and (3.10) for the derivatives of \( w_\tau(z) \) and \( \hat{N}(z) \), it can be verified that

\[
U_{\sigma j, n_j + \nu_j}(z)
\]

\[
= U_{\sigma j, n_j + \nu_j}(\hat{z}) \cdot \left\{ 1 + se^{i\theta} \left[ \frac{(n_j + \nu_j)g_{\sigma j}(\hat{z})}{\hat{z}} + \frac{1}{g_{\sigma j}(\hat{z}) \cdot \hat{z}} \right] + O(s^2) \right\}.
\]

But since \( |U_{\sigma j, n_j + \nu_j}(\hat{z})| = 1 \), it follows from (5.1) and (5.2), on taking moduli, that
\( (5.3) \quad 1 + \frac{s}{\hat{r}} \left[ (n_j + \nu_j) \text{Re} \left( g_{\sigma_j}(\hat{z}) \right) + \text{Re} \left( \frac{1}{g_{\sigma_j}(\hat{z})} \right) \right] + O(s^2) = 1 + O \left( \frac{1}{n_j + \nu_j} \right). \)

As in Sect. 3, we know in general that \( \text{Re} \left( g_{\sigma_j}(z) \right) \) and \( 1/\text{Re} \left( g_{\sigma_j}(z) \right) \) are uniformly bounded at all points of the unit disk not in \( \hat{C}_{\delta, \sigma_j} \). In particular, for any zero \( z \) of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \) in the unit disk not in \( \hat{C}_{\delta, \sigma_j} \), we see from (5.3) that

\[
\frac{s}{\hat{r}} = O \left( \frac{1}{(n_j + \nu_j)^2} \right) \quad (j \to \infty).
\]

But since the arc \( D_{\sigma_j, n_j+\nu_j} \) cannot pass through 0 and since \( \hat{r} \) is bounded above by unity, then

\[
s = O \left( \frac{1}{(n_j + \nu_j)^2} \right),
\]

for any zero \( z \) of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \) not in \( \hat{C}_{\delta, \sigma_j} \), as \( j \to \infty \). Again, as \( |s| \) measures the distance from \( z \) to a particular point, \( \hat{z} \), of \( D_{\sigma_j, n_j+\nu_j} \), then \( \text{dist} \left[ z; D_{\sigma_j, n_j+\nu_j} \right] \leq |s| \) for any zero of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \), in the unit disk not in \( \hat{C}_{\delta, \sigma_j} \), and it follows that

\[
\text{dist} \left[ \left\{ z_{n_j, \nu_j}(k) \right\}_{k=1}^{n_j} \setminus \hat{C}_{\delta, \sigma_j}; D_{\sigma_j, n_j+\nu_j} \right] = O \left( \frac{1}{(n_j + \nu_j)^2} \right) \quad (j \to \infty),
\]

which is the desired result of (2.10) of Theorem 3. \( \square \)

6. Final Comments

The results of the previous sections deal with the convergence of the zeros (and poles) of the Padé approximants \( \{ R_{n_j, \nu_j}((n_j + \nu_j)z) \}_{j=1}^{\infty} \) in relationship to the arcs \( D_{\sigma_j, \infty} \) and \( D_{\sigma_j, n_j+\nu_j} \) outside of the disks \( \hat{C}_{\delta, \sigma_j} \). It is thus natural to ask what the convergence rate of these zeros is in the neighborhood of the points \( z_{\pm, \sigma_j} \), which are explicitly excluded in the results of (2.4) of Theorem 1 and (2.10) of Theorem 3. It turns out that, on applying a result of [9, eq.(1.9)], we also have the following result:

**Theorem 4.** Under the hypothesis of (2.5) of Theorem 1, the Padé approximant \( R_{n_j, \nu_j}((n_j + \nu_j)z) \) has zeros and poles of the form

\[
\pm \frac{1}{(n_j + \nu_j)^{\frac{1}{2}}} + O \left( \frac{1}{(n_j + \nu_j)^{\frac{3}{2}}} \right) \quad (j \to \infty).
\]

The importance of Theorem 4 lies in the fact that (6.1) is valid for any \( \sigma \) with \( 0 < \sigma < \infty \), and Theorem 4 shows that there is a substantially slower convergence of the zeros and poles of \( R_{n_j, \nu_j}((n_j + \nu_j)z) \) to \( D_{\sigma_j, \infty} \) and \( E_{\sigma_j, \infty} \) in neighborhoods of the branch points, \( z_{\pm, \sigma} \), of \( g_{\sigma}(z) \), which are exactly the points which have been excluded with our use of the disks \( \hat{C}_{\delta, \sigma} \) of (2.2). In this sense, (6.1) of Theorem 4 is the analog of result (1.2) of Buckholtz [2]. We conjecture that the results of (6.1) are best possible!

Next, we remark that a careful examination shows that the arcs \( D_{1, \infty} \) and \( E_{1, \infty} \) in Fig. 3 make an angle of \( \pm \pi/3 \) as they meet at the points \( \pm i \). This has been theoretically established in Olver [7, p.336, Fig. 3] for the special case \( \sigma = 1 \), and we remark that
Table 1.

| k | \text{arg}(z_{16,24}(k)) | |z_{16,24}(k)| | |z_{16,24}(k)| |
|---|---|---|---|---|
| 1 | 114.677512621653575 | 0.924518801046913 | 0.923788102974373 |
| 2 | 125.542379165422707 | 0.864754514607011 | 0.884321571680979 |
| 3 | 134.974717991547015 | 0.825049238737003 | 0.824718583665449 |
| 4 | 143.729353507749157 | 0.796647826203171 | 0.796364645259408 |
| 5 | 152.091848989003633 | 0.776141233642082 | 0.775887263220706 |
| 6 | 160.211234669319259 | 0.761805526883314 | 0.761569325056279 |
| 7 | 168.181077469080952 | 0.752680657788123 | 0.752454813408059 |
| 8 | 176.09064193694423 | 0.748237220620139 | 0.748016178753669 |

this same angle appears also in Figs. 2 and 4. We similarly conjecture that this same angle will appear in all cases where $0 < \sigma < \infty$, because of the fact that (6.1) of Theorem 4 is valid for all $0 < \sigma < \infty$.

Finally, because the results of Figs. 5–7 are almost too good to believe, we consider again, as in Fig. 5, the actual zeros $\{z_{16,24}(k)\}_{k=1}^N$ of $R_{16,24}(40z)$ in the upper half-plane, ordered by increasing argument. In columns 2 and 3 of Table 1 above, we give respectively the arguments and moduli of these eight zeros. Then, $\tilde{z}_{16,24}(k)$ is defined (as in the proof of Theorem 3 in Sect. 5) as the point on the arc $D_{3/2,40}$ having the same argument as $z_{16,24}(k)$, and the last column of Table 1 gives the corresponding moduli of the points $\{\tilde{z}_{16,24}(k)\}_{k=1}^N$. The differences of these corresponding moduli in columns 3 and 4 vary between $7.3 \cdot 10^{-4}$ and $2.2 \cdot 10^{-4}$, which is why, up to plotting accuracy, the zeros of $R_{16,24}(40z)$ appear to lie on the arc $D_{3/2,40}$.

References


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