

GERŠGORIN-TYPE EIGENVALUE INCLUSION THEOREMS AND THEIR SHARPNESS*

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Dedicated to Professor John Todd, on the occasion of his 90th birthday, May 16, 2001.

Abstract. Here, we investigate the relationships between $\mathcal{G}(A)$, the union of Geršgorin disks, $\mathcal{K}(A)$, the union of Brauer ovals of Cassini, and $\mathcal{B}(A)$, the union of Brualdi lemniscate sets, for eigenvalue inclusions of an $n \times n$ complex matrix A . If $\sigma(A)$ denotes the spectrum of A , we show here that

$$\sigma(A) \subseteq \mathcal{B}(A) \subseteq \mathcal{K}(A) \subseteq \mathcal{G}(A)$$

is valid for any weakly irreducible $n \times n$ complex matrix A with $n \geq 2$. Further, it is evident that $\mathcal{B}(A)$ can contain the spectra of related $n \times n$ matrices. We show here that the spectra of these related matrices can fill out $\mathcal{B}(A)$. Finally, if $\mathcal{G}^{\mathcal{R}}(A)$ denotes the minimal Geršgorin set for A , we show that

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A).$$

Key words. Geršgorin disks, Brauer ovals of Cassini, Brualdi lemniscate sets, minimal Geršgorin sets.

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1. Geršgorin Disks and Ovals of Cassini. For any $n \geq 2$, let A be any $n \times n$ complex matrix (written $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$), and let $\sigma(A)$ denote its **spectrum** (i.e., $\sigma(A) := \{\lambda \in \mathbb{C} : \det[A - \lambda I_n] = 0\}$). A familiar result of Geršgorin [3] is that if

$$(1.1) \quad G_i(A) := \left\{ z \in \mathbb{C} : |z - a_{i,i}| \leq r_i(A) := \sum_{\substack{j=1 \\ j \neq i}}^n |a_{i,j}| \right\} \quad (1 \leq i \leq n)$$

denotes the i -th **Geršgorin disk** for A , then the union of these n Geršgorin disks contains all eigenvalues of A :

$$(1.2) \quad \sigma(A) \subseteq \mathcal{G}(A) := \bigcup_{i=1}^n G_i(A).$$

A less familiar result of Brauer [1] is that if

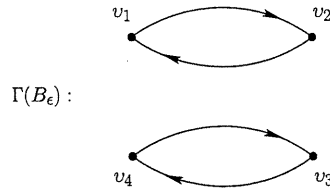
$$(1.3) \quad K_{i,j}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i(A) \cdot r_j(A)\} \quad (1 \leq i, j \leq n; i \neq j)$$

denotes the (i, j) -th **Brauer oval of Cassini** for A , then similarly

$$(1.4) \quad \sigma(A) \subseteq \mathcal{K}(A) := \bigcup_{\substack{i,j=1 \\ i \neq j}}^n K_{i,j}(A),$$

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We note that the eigenvalue inclusion of Brualdi's Theorem C, applied to a weakly irreducible matrix $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ with $n \geq 2$, now depends on all the quantities of

$$(2.8) \quad \{a_{i,i}\}_{i=1}^n, \{r_i(A)\}_{i=1}^n, \text{ and } \mathcal{C}(A).$$

As (2.8) requires *more* information from the matrix A , in order to form the Brualdi lemniscate set $\mathcal{B}(A)$, then is required by the Brauer ovals of Cassini $\mathcal{K}(A)$ or the Geršgorin disks $\mathcal{G}(A)$, one might expect that $\mathcal{B}(A)$ is a set, which is *no larger*, in the complex plane, than $\mathcal{K}(A)$ or $\mathcal{G}(A)$. This will be shown to be true in Theorem 1 of the next section. In addition, one can ask, in the spirit of Theorem B, if the union of the spectra of all matrices, which match the data of (2.8), fills out the Brualdi lemniscate set $\mathcal{B}(A)$ of (2.7). This will be precisely answered in Theorem 2 of Section 4.

3. Comparison of Brauer's Ovals of Cassini and Brualdi's Lemniscate Sets. Our new result here is very much in the spirit of the proof of Theorem A. (See also [6].)

Theorem 1. *For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, which is weakly irreducible, then, with the definitions of (1.4) and (2.7),*

$$(3.1) \quad \mathcal{B}(A) \subseteq \mathcal{K}(A).$$

Remark. This establishes that the Brualdi lemniscate set for a matrix A is always no larger than the union of the Brauer ovals of Cassini for this matrix.

Proof. Consider any circuit γ in $\mathcal{C}(A)$. If this circuit has length two, i.e., $\gamma = (i_1, i_2)$, where $i_3 = i_1$, it follows from (2.7) that

$$(3.2) \quad \mathcal{B}_\gamma(A) := \{z \in \mathbb{C} : |z - a_{i_1, i_1}| \cdot |z - a_{i_2, i_2}| \leq r_{i_1}(A) \cdot r_{i_2}(A)\},$$

which, from (1.3), is exactly $\mathcal{K}_{i_1, i_2}(A)$, i.e.,

$$(3.3) \quad \mathcal{B}_\gamma(A) = \mathcal{K}_{i_1, i_2}(A).$$

Next, assume that γ has length $p > 2$, i.e., $\gamma = (i_1, i_2, \dots, i_p)$, where $i_{p+1} = i_1$. Since A is weakly irreducible, each vertex of $\Gamma(A)$ has a circuit passing through it, so that $r_\ell(A) > 0$ for each ℓ in N . From (2.7), we define

$$(3.4) \quad \mathcal{B}_\gamma(A) := \left\{ z \in \mathbb{C} : \prod_{j=1}^p |z - a_{i_j, i_j}| \leq \prod_{j=1}^p r_{i_j}(A) \right\}.$$

Let z be any point of $\mathcal{B}_\gamma(A)$. On squaring the inequality in (3.4), we have

$$|z - a_{i_1, i_1}|^2 \cdot |z - a_{i_2, i_2}|^2 \cdots |z - a_{i_p, i_p}|^2 \leq r_{i_1}^2(A) \cdot r_{i_2}^2(A) \cdots r_{i_p}^2(A).$$

As these $r_{i_j}(A)$'s are all positive, we can equivalently express the above inequality as

$$(3.5) \quad \left(\frac{|z - a_{i_1, i_1}| \cdot |z - a_{i_2, i_2}|}{r_{i_1}(A) \cdot r_{i_2}(A)} \right) \cdot \left(\frac{|z - a_{i_2, i_2}| \cdot |z - a_{i_3, i_3}|}{r_{i_2}(A) \cdot r_{i_3}(A)} \right) \cdots \cdot \left(\frac{|z - a_{i_p, i_p}| \cdot |z - a_{i_1, i_1}|}{r_{i_p}(A) \cdot r_{i_1}(A)} \right) \leq 1.$$

As the factors on the left of (3.5) cannot all exceed unity, then at least one of the factors is at most unity. Hence, there is an ℓ with $1 \leq \ell \leq p$ such that

$$|z - a_{i_\ell, i_\ell}| \cdot |z - a_{i_{\ell+1}, i_{\ell+1}}| \leq r_{i_\ell}(A) \cdot r_{i_{\ell+1}}(A)$$

(where if $\ell = p$, then $i_{\ell+1} = i_1$). But from the definition in (1.3), we see that $z \in K_{i_\ell, i_{\ell+1}}(A)$, and, as a consequence, it follows that

$$(3.6) \quad \mathcal{B}_\gamma(A) \subseteq \bigcup_{j=1}^p K_{i_j, i_{j+1}}(A) \quad (\text{where } i_{p+1} = i_1).$$

Thus, from (2.7) and (3.6),

$$\mathcal{B}(A) := \bigcup_{\gamma \in \mathcal{C}(A)} \mathcal{B}_\gamma(A) \subseteq \bigcup_{\substack{i, j=1 \\ i \neq j}}^n K_{i, j}(A) := \mathcal{K}(A),$$

the desired result of (3.1). \square

We next show that there are many cases where equality holds in (3.1). Consider any matrix $A = [a_{i, j}] \in \mathbb{T}^{n \times n}$, with $n > 2$, for which every non-diagonal entry $a_{i, j}$ of A is nonzero. The matrix A is then clearly irreducible, and hence, weakly irreducible. Moreover, every partition, of length ≥ 2 , of \mathcal{P}_n of (2.5) can be associated with a circuit γ of $\mathcal{C}(A)$, so that the total number of these circuits is, from (2.5),

$$(3.7) \quad \sum_{k=2}^n \binom{n}{k} (k-1)!.$$

In this case, as each Brauer oval of Cassini $K_{i, j}(A)$, $i \neq j$, corresponds to a circuit (of length 2) in $\mathcal{B}(A)$, it follows from (2.7) that $\mathcal{K}(A) \subseteq \mathcal{B}(A)$, but as the reverse inclusion holds in (3.1), then

$$(3.8) \quad \mathcal{B}(A) = \mathcal{K}(A).$$

In other words, the Brauer set $\mathcal{B}(A)$ need not, in general, be a proper subset of the Brauer ovals of Cassini $\mathcal{K}(A)$. We remark that for a 10×10 complex matrix A , all of whose off-diagonal entries are nonzero, there are, from (3.7), 1,112,073 distinct circuits in $\mathcal{C}(A)$. But because of (3.8), only $\binom{10}{2} = 45$ of these circuits, corresponding to the Brauer ovals of Cassini, are needed to determine $\mathcal{B}(A)$.

4. The Sharpness of Brualdi Lemniscate Sets. Given $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, with $n \geq 2$ for which A is weakly irreducible, we have from (2.7) of Theorem C that

$$(4.1) \quad \sigma(A) \subseteq \mathcal{B}(A),$$

where the associated Brualdi lemniscate set $\mathcal{B}(A)$ is determined, in (2.7), from the quantities

$$(4.2) \quad \{a_{i,i}\}_{i=1}^n, \quad \{r_i(A)\}_{i=1}^n, \quad \text{and } \mathcal{C}(A).$$

It is again evident that any matrix $B = [b_{i,j}] \in \mathbb{C}^{n \times n}$, having the identical quantities of (4.2), has its eigenvalues also in $\mathcal{B}(A)$, i.e., with notations similar to (1.11) and (1.12), if

$$(4.3) \quad \Omega_{\mathcal{B}}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i}, r_i(B) = r_i(A), 1 \leq i \leq n, \\ \text{and } \mathcal{C}(B) = \mathcal{C}(A)\},$$

where $\sigma(\Omega_{\mathcal{B}}(A)) := \bigcup_{B \in \Omega_{\mathcal{B}}(A)} \sigma(B)$, and if

$$(4.4) \quad \hat{\Omega}_{\mathcal{B}}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i}, 0 < r_i(B) \leq r_i(A), 1 \leq i \leq n, \\ \text{and } \mathcal{C}(B) = \mathcal{C}(A)\},$$

where $\sigma(\hat{\Omega}_{\mathcal{B}}(A)) := \bigcup_{B \in \hat{\Omega}_{\mathcal{B}}(A)} \sigma(B)$, it follows, in analogy with (1.14), that

$$(4.5) \quad \sigma(\Omega_{\mathcal{B}}(A)) \subseteq \sigma(\hat{\Omega}_{\mathcal{B}}(A)) \subseteq \mathcal{B}(A).$$

(We note, since A is weakly irreducible, that $r_i(A) > 0$ for all $1 \leq i \leq n$.)

It is natural to ask if equality can hold throughout in (4.5). The answer, in general, is **no**, as the following simple example shows. Consider the matrix

$$(4.6) \quad D = \begin{bmatrix} +1 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & +1 \end{bmatrix},$$

so that

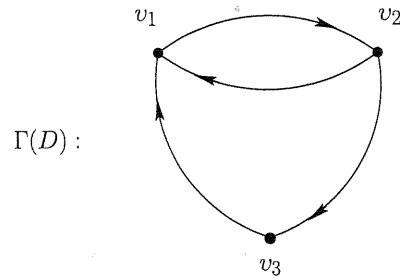
$$r_1(D) = r_2(D) = 1, \quad \text{and } r_3(D) = \frac{1}{2}.$$

The directed graph $\Gamma(D)$ is then so that D is irreducible, and the circuit set of D is

$$\mathcal{C}(D) = (1, 2) \cup (1, 2, 3).$$

Now, any matrix E in $\Omega_{\mathcal{B}}(D)$ can be expressed, from (4.3), as

$$(4.7) \quad E = \begin{bmatrix} 1 & e^{i\theta_1} & 0 \\ (1-s)e^{i\theta_2} & -1 & se^{i\theta_3} \\ \frac{1}{2}e^{i\theta_4} & 0 & 1 \end{bmatrix},$$



where s satisfies $0 < s < 1$ and where $\{\theta_i\}_{i=1}^4$ are any real numbers. (Note that letting $s = 0$ or $s = 1$ in (4.7) does *not* preserve the circuit sets of $\mathcal{C}(D)$.) With $\gamma_1 := (1, 2)$ and $\gamma_2 := (1, 2, 3)$, we see that

$$(4.8) \quad \begin{cases} \mathcal{B}_{\gamma_1}(D) = \{z \in \mathbb{C} : |z - 1| \cdot |z + 1| \leq 1\}, \text{ and} \\ \mathcal{B}_{\gamma_2}(D) = \{z \in \mathbb{C} : |z - 1|^2 \cdot |z + 1| \leq 1/2\}, \end{cases}$$

where the set $\mathcal{B}_{\gamma_2}(D)$ consists of two disjoint components. These sets are shown in Figure 1.

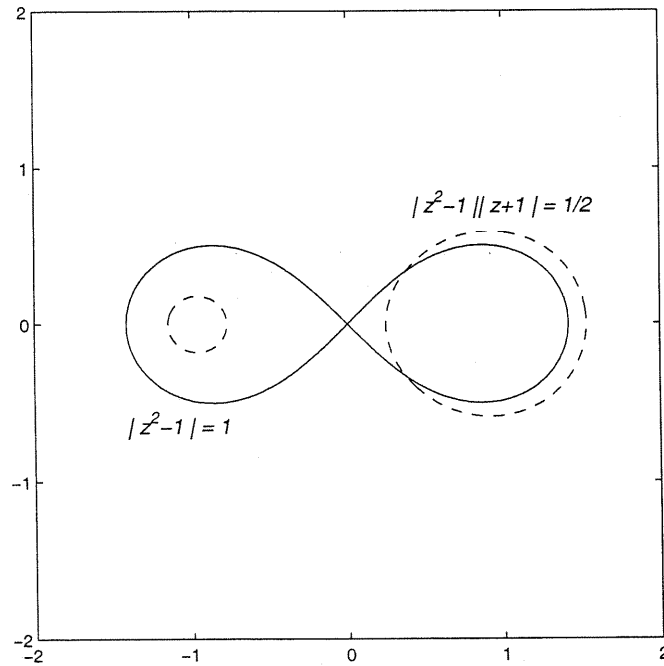


Figure 1

It can be seen, from (4.8) and from Figure 1, that $z = 0$ is a boundary point of the compact sets $\mathcal{B}_{\gamma_1}(D)$ and $\mathcal{B}(D) := \mathcal{B}_{\gamma_1}(D) \cup \mathcal{B}_{\gamma_2}(D)$. Suppose that we can find an s with $0 < s < 1$ and real values of $\{\theta_i\}_{i=1}^4$ for which an associated matrix E of (4.7) has eigenvalue 0. This implies that $\det E = 0$, which, by direct calculations with (4.7), gives

$$0 = \det E = -1 - (1-s)e^{i(\theta_1+\theta_2)} + \frac{1}{2}se^{i(\theta_1+\theta_3+\theta_4)}, \text{ or}$$

$$(4.9) \quad 1 = \left\{ -(1-s)e^{i(\theta_1+\theta_2)} + \frac{1}{2}se^{i(\theta_1+\theta_3+\theta_4)} \right\}.$$

But as $0 < s < 1$, the right side of (4.9) is in modulus at most

$$(1-s) + \frac{1}{2}s = \frac{2-s}{2} < 1,$$

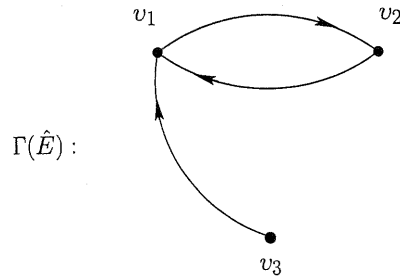
so that $\det E \neq 0$ for any E in $\Omega_B(D)$, i.e., $0 \notin \sigma(\Omega_B(D))$. A similar argument shows (cf. (4.4)) that $0 \notin \sigma(\hat{\Omega}_B(D))$. But as $0 \in \mathcal{B}(D)$, we have

$$(4.10) \quad \sigma(\Omega_B(D)) \subseteq \sigma(\hat{\Omega}_B(D)) \subsetneq \mathcal{B}(A).$$

But, in order to achieve equality in (4.10), suppose that we *allow* s to be zero, noting from (4.8) that the parameter s plays *no role* in $\mathcal{B}(D) = \mathcal{B}_{\gamma_1}(D) \cup \mathcal{B}_{\gamma_2}(D)$. Then, on setting $s = 0$ in (4.7), the matrix E of (4.7) becomes

$$(4.11) \quad \hat{E} = \begin{bmatrix} 1 & e^{i\theta_1} & 0 \\ e^{i\theta_2} & -1 & 0 \\ \frac{1}{2}e^{i\theta_4} & 0 & 1 \end{bmatrix},$$

and on choosing $\theta_1 = 0$ and $\theta_2 = \pi$, then $z = 0$ is an eigenvalue of \hat{E} , where \hat{E} is the *limit* of matrices E in (4.7) when $s \downarrow 0$. We note that the directed graph of $\Gamma(\hat{E})$ is



so that $\mathcal{C}(\hat{E}) \neq \mathcal{C}(D)$, but \hat{E} remains an element of $\Omega(A)$ of (1.11).

This example suggests that we consider the **closures** of the sets $\Omega_B(A)$ and $\hat{\Omega}_B(A)$ of (4.3) and (4.4), where $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is weakly irreducible:

$$(4.12) \quad \overline{\Omega}_B(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : \text{there is a sequence of matrices } \{E_j\}_{j=1}^{\infty} \text{ in } \Omega_B(A), \text{ for which } B = \lim_{j \rightarrow \infty} E_j\}.$$

and

$$(4.13) \quad \overline{\hat{\Omega}}_{\mathcal{B}}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : \text{there is a sequence of matrices } \{E_j\}_{j=1}^{\infty} \text{ in } \hat{\Omega}_{\mathcal{B}}(A), \text{ for which } B = \lim_{j \rightarrow \infty} E_j.\}$$

This brings us to the new result of

Theorem 2. For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, which is weakly irreducible, then

$$(4.14) \quad \partial \mathcal{B}(A) \subseteq \sigma(\overline{\Omega}_{\mathcal{B}}(A)) \subseteq \sigma(\overline{\hat{\Omega}}_{\mathcal{B}}(A)) = \mathcal{B}(A),$$

i.e., each boundary point of $\mathcal{B}(A)$ is an eigenvalue of some matrix in $\overline{\Omega}_{\mathcal{B}}(A)$, and each point of $\mathcal{B}(A)$ is an eigenvalue of some matrix in $\overline{\hat{\Omega}}_{\mathcal{B}}(A)$.

Remark. This establishes the sharpness of the Brualdi set $\mathcal{B}(A)$ for the matrix A , as the final equality in (4.14) gives that the spectra of matrices in $\hat{\Omega}_{\mathcal{B}}(A)$ are dense in $\mathcal{B}(A)$.

Proof. Since $\sigma(\Omega_{\mathcal{B}}(A)) \subseteq \sigma(\hat{\Omega}_{\mathcal{B}}(A))$ from (4.5), it follows that their closures of (4.12) and (4.13) necessarily satisfy $\sigma(\overline{\Omega}_{\mathcal{B}}(A)) \subseteq \sigma(\overline{\hat{\Omega}}_{\mathcal{B}}(A))$, giving the middle inclusion of (4.14). It suffices to establish the first inclusion and the final equality in (4.14).

Consider any circuit γ in $\mathcal{C}(A)$. From our discussion in Section 2, we can express γ as an element of \mathcal{P}_n of (2.5), i.e.,

$$(4.15) \quad \gamma = (i_1, i_2, \dots, i_p) \text{ where } i_{p+1} := i_1, \text{ and where } 2 \leq p \leq n.$$

Without loss of generality, we can assume, after a suitable permutation of the rows and columns of A , that

$$(4.16) \quad \gamma = (1, 2, \dots, p),$$

noting that this permutation leaves unchanged the collection of diagonal entries, row sums, and circuits of A . This permuted matrix, also called A , then has the partitioned form

$$(4.17) \quad A = \left[\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,p} & a_{1,p+1} & \cdots & a_{1,n} \\ \vdots & & & & & \vdots \\ a_{p,1} & & a_{p,p} & a_{p,p+1} & & a_{p,n} \\ \hline a_{p+1,1} & & a_{p+1,p} & a_{p+1,p+1} & & a_{p+1,n} \\ \vdots & & & & & \vdots \\ a_{n,1} & \cdots & a_{n,p} & a_{n,p+1} & \cdots & a_{n,n} \end{array} \right] = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array} \right],$$

where the matrices $A_{1,2}$, $A_{2,1}$, and $A_{2,2}$ are not present in (4.17) if $p = n$. Our aim below is to construct a special matrix $B(t) = [b_{i,j}(t)] \in \mathbb{C}^{n \times n}$, whose entries depend continuously on the parameter t in $[0, 1]$, such that

$$(4.18) \quad \begin{cases} b_{i,i}(t) = a_{i,i}, r_i(B(t)) = r_i(A), \text{ for all } 1 \leq i \leq n, \text{ and all } 0 \leq t \leq 1, \text{ and} \\ C(B(t)) = C(A) \text{ for all } 0 < t \leq 1. \end{cases}$$

To this end, write

$$(4.19) \quad B(t) := \left[\begin{array}{c|c} B_{1,1}(t) & B_{1,2}(t) \\ \hline A_{2,1} & A_{2,2} \end{array} \right],$$

i.e., the rows $p + 1 \leq \ell \leq n$ of $B(t)$ are exactly those of A , and are independent of t . We note from (4.16) that

$$(4.20) \quad a_{1,2} \cdot a_{2,3} \cdots a_{p-1,p} \cdot a_{p,1} \neq 0,$$

and the rows of $B(t)$ are then defined, for all $t \in [0, 1]$, by

$$(4.21) \quad \begin{cases} b_{i,i}(t) := a_{i,i} \text{ for all } 1 \leq i \leq p; \\ |b_{i,i+1}(t)| := (1-t)r_i(A) + t|a_{i,i+1}|, \text{ and } |b_{i,j}(t)| := t|a_{i,j}| \quad (j \neq i, i+1), \\ \text{for all } 1 \leq i < p; \\ |b_{p,1}(t)| := (1-t)r_p(A) + t|a_{p,1}|, \text{ and } |b_{p,j}(t)| := t|a_{p,j}| \quad (\text{all } j \neq 1, p). \end{cases}$$

It is evident that the entries of $B(t)$ are all continuous in the variable t of $[0, 1]$. Moreover, $B(t)$ and A have the same diagonal entries and the same row sums for all $0 \leq t \leq 1$, and, as $a_{i,j} \neq 0$ implies $b_{i,j}(t) \neq 0$ for all $0 < t \leq 1$, then $B(t)$ and A have the same circuits in their directed graphs for all $0 < t \leq 1$. As A is weakly irreducible by hypothesis, it follows that $B(t)$ is weakly irreducible for all $0 < t \leq 1$. Also, from (4.3), $B(t) \in \Omega_B(A)$ for all $0 < t \leq 1$, and from (4.12), $B(0) \in \overline{\Omega_B(A)}$. Hence, from (4.5),

$$(4.22) \quad \sigma(B(t)) \subseteq \mathcal{B}(A) \text{ for all } 0 < t \leq 1.$$

But as $\mathcal{B}(A)$ is a closed set from (2.7), and as the eigenvalues of $B(t)$ are continuous functions of t , for $0 \leq t \leq 1$, we further have, for the limiting case $t = 0$, that

$$(4.23) \quad \sigma(B(0)) \subseteq \mathcal{B}(A),$$

where, from the definitions in (4.21),

$$B(0) = \left[\begin{array}{c|c} B_{1,1}(0) & O \\ \hline A_{2,1} & A_{2,2} \end{array} \right],$$

with

$$(4.24) \quad B_{1,1}(0) = \begin{bmatrix} a_{1,1} & r_1(A)e^{i\theta_1} & & & \\ & a_{2,2} & r_2(A)e^{i\theta_2} & & \\ & & \ddots & \ddots & \\ & & & a_{p-1,p-1} & r_{p-1}(A)e^{i\theta_{p-1}} \\ r_p(A)e^{i\theta_p} & & & & a_{p,p} \end{bmatrix}.$$

We note from (4.21) that the nondiagonal entries in the first p rows of $B(t)$ are defined only in terms of their moduli, which allows us to fix the arguments of certain nondiagonal nonzero

entries of $B_{1,1}(0)$ through the factors $\{e^{i\theta_j}\}_{j=1}^p$ where the $\{\theta_j\}_{j=1}^p$ are all real. (These factors appear in $B_{1,1}(0)$ of (4.24).) The partitioned form of $B(0)$ gives us that

$$(4.25) \quad \sigma(B(0)) = \sigma(B_{1,1}(0)) \cup \sigma(A_{2,2}),$$

and, from the special cyclic-like form of $B_{1,1}(0)$ in (4.24), it is easily seen that each eigenvalue λ of $B_{1,1}(0)$ satisfies

$$(4.26) \quad \prod_{i=1}^p |\lambda - a_{i,i}| = \prod_{i=1}^p r_i(A),$$

for any real choices of $\{\theta_j\}_{j=1}^p$ in (4.24). But (4.26), when coupled with the definition in (3.4), immediately gives us that $\lambda \in \partial\mathcal{B}_\gamma(A)$, and, as all different choices of the real numbers $\{\theta_j\}_{j=1}^p$, in $B_{1,1}(0)$ of (4.24), give eigenvalues of $B_{1,1}(0)$ which cover the *entire boundary* of $\mathcal{B}_\gamma(A)$, we have

$$(4.27) \quad \bigcup_{\theta_1, \dots, \theta_p \text{ real}} \sigma(B_{1,1}(0)) = \partial\mathcal{B}_\gamma(A).$$

This can be used as follows. Let z be any boundary point of $\mathcal{B}(A)$ of (2.7). As $\mathcal{B}(A)$ is the union of a finite number of closed sets $\mathcal{B}_\gamma(A)$, this implies that there is a circuit γ of $\mathcal{C}(A)$ with $z \in \partial\mathcal{B}_\gamma(A)$, where $\mathcal{B}_\gamma(A)$ is defined in (3.4). As the result of (4.27) is valid for any γ of $\mathcal{C}(A)$, then each boundary point z of $\mathcal{B}(A)$ is an eigenvalue of some matrix in $\overline{\Omega}_B(A)$ of (4.12), i.e.,

$$(4.28) \quad \partial\mathcal{B}(A) \subseteq \sigma(\overline{\Omega}_B(A)),$$

which is the desired first inclusion of (4.14).

To investigate how the eigenvalues of $\overline{\Omega}_B(A)$ of (4.13) fill out $\mathcal{B}(A)$, we make a small change in the definition of the matrix $B(t)$ of (4.18) and (4.21). Let $\{\tau_i\}_{i=1}^p$ be any positive numbers such that

$$(4.29) \quad 0 < \tau_i \leq r_i(A) \quad (1 \leq i \leq p),$$

and let $\tilde{B}(t) = [\tilde{b}_{i,j}(t)] \in \mathbb{C}^{n \times n}$ have the same partitioned form as $B(t)$ of (4.19), but with (4.21) replaced by

$$(4.30) \quad \begin{cases} \tilde{b}_{i,i}(t) := a_{i,i} \text{ for all } 1 \leq i \leq p, \text{ and} \\ |\tilde{b}_{i,j}(t)| := \frac{\tau_i}{r_i(A)} |b_{i,j}(t)| \quad (j \neq i), \text{ for } 1 \leq i \leq p. \end{cases}$$

Then, $\tilde{B}(t)$ and A have the same diagonal entries, the row sums of $\tilde{B}(t)$ now satisfy $r_j(\tilde{B}(t)) = \tau_j$ for all $1 \leq j \leq p$, all $0 \leq t \leq 1$, and $\tilde{B}(t)$ and A have the same circuits for all $0 < t \leq 1$. From (4.4), $\tilde{B}(t) \in \overline{\Omega}_B(A)$ for all $0 < t \leq 1$, and from (4.13), $\tilde{B}(0) \in \overline{\Omega}_B(A)$. In analogy with (4.23), we have

$$\tilde{B}(0) = \left[\begin{array}{c|c} \tilde{B}_{1,1}(0) & O \\ \hline A_{2,1} & A_{2,2} \end{array} \right],$$

with

$$(4.31) \quad \tilde{B}_{1,1}(0) = \left[\begin{array}{cccc} a_{1,1} & \tau_1 e^{i\theta_1} & & \\ & a_{2,2} & \tau_2 e^{i\theta_2} & \\ & & \ddots & \ddots \\ \tau_p e^{i\theta_p} & & a_{p-1,p-1} & \tau_{p-1} e^{i\theta_{p-1}} \\ & & & a_{p,p} \end{array} \right],$$

where

$$(4.32) \quad \sigma(\tilde{B}(0)) = \sigma(\tilde{B}_{1,1}(0)) \cup \sigma(A_{2,2}).$$

It similarly follows that any eigenvalue λ of $\tilde{B}_{1,1}(0)$ in (4.31) now satisfies

$$(4.33) \quad \prod_{j=1}^p |\lambda - a_{i,j}| = \prod_{i=1}^p \tau_i,$$

for any choice of the real numbers $\{\theta_j\}_{j=1}^p$ in $\tilde{B}_{1,1}(0)$ of (4.31). Using the fact that $\{\tau_i\}_{i=1}^p$ are any numbers satisfying (4.29) and that $\{\theta_i\}_{i=1}^p$ are any real numbers, it follows from (3.4) and closure considerations that all the eigenvalues of $\tilde{B}_{1,1}(0)$ fill out $\mathcal{B}_\gamma(A)$, i.e.,

$$(4.34) \quad \bigcup_{\tau'_i s, \theta'_i s} \sigma(\tilde{B}_{1,1}(0)) = \mathcal{B}_\gamma(A).$$

As this holds for any $\gamma \in \mathcal{C}(A)$, where $\tilde{B}(0) \in \overline{\Omega}_B(A)$, then

$$(4.35) \quad \sigma(\overline{\Omega}_B(A)) = \mathcal{B}(A),$$

the desired final result of (4.14). \square

5. A Comparison of Minimal Geršgorin Sets and Brualdi Lemniscate Sets. Given $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ and $\mathbf{x} = [x_1, x_2, \dots, x_n]^T > \mathbf{0}$ in \mathbb{R}^n , then with $X := \text{diag}[x_1, x_2, \dots, x_n]$, we have that $X^{-1}AX = \begin{bmatrix} a_{i,j} x_j \\ x_i \end{bmatrix}$, where, since A and $X^{-1}AX$ are similar matrices, $\sigma(A) = \sigma(X^{-1}AX)$. On setting

$$(5.1) \quad r_i^{\mathbf{x}}(A) := \sum_{\substack{j=1 \\ j \neq i}}^n \frac{|a_{i,j}| x_j}{x_i} \quad (\text{all } i \in N),$$

then Geršgorin Theorem, applied to $X^{-1}AX$, gives

$$(5.2) \quad \sigma(A) = \sigma(X^{-1}AX) \subseteq \bigcup_{i=1}^n \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i^x(A)\} =: \mathcal{G}^x(A).$$

As this eigenvalue inclusion holds for each $\mathbf{x} = [x_1, x_2, \dots, x_n]^T > \mathbf{0}$ in \mathbb{R}^n , we have

$$(5.3) \quad \sigma(A) \subseteq \bigcap_{\mathbf{x} > \mathbf{0}} \mathcal{G}^x(A) =: \mathcal{G}^{\mathcal{R}}(A),$$

and $\mathcal{G}^{\mathcal{R}}(A)$ is called (cf. [5]) the **minimal Geršgorin set** for A , relative to the collection of all weighted row sums $\{r_i^x(A)\}_{i=1}^n$. As $\mathcal{G}^x(A)$ is, for each $\mathbf{x} > \mathbf{0}$, the union of n closed disks in the complex plane \mathbb{C} , then $\mathcal{G}^x(A)$ is compact, as is $\mathcal{G}^{\mathcal{R}}(A)$.

Another natural question is how the Braualdi lemniscate set $\mathcal{B}(A)$ of (2.7), a compact set in \mathbb{C} , compares with the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ of (5.3), for every $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$. The first apparent difference is that the Braualdi lemniscate set $\mathcal{B}(A)$ requires A to be weakly irreducible, whereas the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ does not make this restriction. But, if we assume that $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ is weakly reducible, we can compare the associated sets in the complex plane. Given $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, we consider the following set of matrices, associated with A :

$$(5.4) \quad \Delta(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i} \text{ and } |b_{i,j}| \leq |a_{i,j}| \text{ for all } i \neq j; i, j \in N\}.$$

Lemma 3. For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, which is weakly irreducible, then (cf. (4.13))

$$(5.5) \quad \Delta(A) \subseteq \overline{\Omega}_{\mathcal{B}}(A).$$

Proof. We first note that as A is weakly irreducible, then $r_i(A) > 0$ for all $i \in N$. Consider any matrix $B = [b_{i,j}]$ in $\Delta(A)$, so that from (5.4),

$$(5.6) \quad r_i(B) \leq r_i(A) \quad \text{for all } i \in N.$$

Set $S_i(A) := \{j \in N : j \neq i \text{ and } |a_{i,j}| > 0\}$, for each $i \in N$. Then, $S_i(A) \neq \emptyset$ for any $i \in N$, since A is weakly irreducible. If there is a j in $S_i(A)$ for which $b_{i,j} = 0$, we note that

$$(5.7) \quad r_i(B) < r_i(A).$$

Then, for a fixed $\epsilon > 0$, replace this (i, j) -th entry of B by any number having modulus ϵ , and do the same for every k in $S_i(A)$ for which $b_{i,k} = 0$, leaving the remaining entries in this i -th row, of B , unchanged. On carrying out this procedure for all rows of the matrix B , a matrix $B(\epsilon)$, in $\mathbb{C}^{n \times n}$, is created, whose entries are continuous in the parameter ϵ , and for which the circuit set $\mathcal{C}(B(\epsilon))$ of $B(\epsilon)$ is identical with the circuit set $\mathcal{C}(A)$ of A , for each

$\epsilon > 0$. In addition, because of the strict inequality in (5.7) whenever $b_{i,j} = 0$ with $j \in S_i(A)$, it follows, for all $\epsilon > 0$ sufficiently small, that

$$(5.8) \quad r_i(B(\epsilon)) \leq r_i(A) \quad \text{for all } i \in N.$$

Hence from (4.4), $B(\epsilon) \in \hat{\Omega}_B(A)$ for all $\epsilon > 0$, sufficiently small. As such, we see from the definition in (4.13) that $B(0) = B \in \overline{\hat{\Omega}}_B(A)$, and as this holds for any $B \in \Delta(A)$, the inclusion of (5.5) is valid. \square

This brings us to the following new result which is both surprising and simple.

Theorem 4. For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, which is weakly irreducible, then

$$(5.9) \quad \mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A).$$

Remark. The word “minimal” in the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ seems to be appropriate!

Proof. It is known from [5] that $\mathcal{G}^{\mathcal{R}}(A)$ of (5.3) satisfies

$$(5.10) \quad \mathcal{G}^{\mathcal{R}}(A) = \sigma(\Delta(A)),$$

and as $\Delta(A) \subseteq \overline{\hat{\Omega}}_B(A)$ from (5.5) of Lemma 3, then $\sigma(\Delta(A)) \subseteq \sigma(\overline{\hat{\Omega}}_B(A))$. But as $\sigma(\overline{\hat{\Omega}}_B(A)) = \mathcal{B}(A)$ from (4.14) of Theorem 2, then these inclusions give that

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A),$$

the desired result of (5.9). \square

6. An Example. To illustrate the above results, consider the matrix

$$(6.1) \quad F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \frac{1}{2} & i & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -i \end{bmatrix}.$$

Then, F is irreducible, with $\mathcal{C}(F) = (1, 2) \cup (1, 2, 3, 4)$, with row sums $r_i(F) = 1$ for all $1 \leq i \leq 4$. In this case, we have from (2.7) that $\mathcal{B}(F)$ consists of the union of the two closed sets

$$(6.2) \quad \begin{cases} \mathcal{B}_{\gamma_1}(F) := \{z \in \mathbb{C} : |z - 1| \cdot |z - i| \leq 1\} = K_{1,2}(A), \text{ and} \\ \mathcal{B}_{\gamma_2}(F) := \{z \in \mathbb{C} : |z^4 - 1| \leq 1\}. \end{cases}$$

These sets are shown in Figure 2, where $\mathcal{B}_{\gamma_2}(F)$ has the shape of a four-leaf clover.

Next, any matrix h in $\hat{\Omega}_B(F)$ can be expressed from (4.4) as

$$(6.3) \quad H = \begin{bmatrix} 1 & \tau_1 e^{i\theta_1} & 0 & 0 \\ \tau_2 s e^{i\theta_2} & i & \tau_3 (1 - s) e^{i\theta_3} & 0 \\ 0 & 0 & -1 & \tau_4 e^{i\theta_4} \\ \tau_5 e^{i\theta_5} & 0 & 0 & -i \end{bmatrix},$$

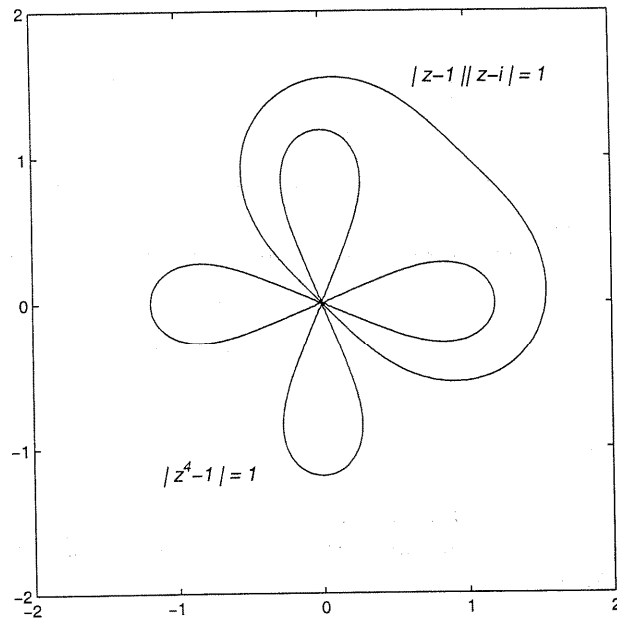


Figure 2

where $0 < \tau_i \leq 1$ and $0 \leq \theta_i \leq 2\pi$ for all $1 \leq i \leq 5$ and $0 < s < 1$. To show how the eigenvalues of H fill out $\mathcal{B}(F)$, we take random numbers $\{\tau_i\}_{i=1}^5$, random numbers $\{\theta_i\}_{i=1}^5$ from $[0, 2\pi)$, and a random number s from $(0, 1)$, and the eigenvalues of all these matrices, are plotted in Figure 3. Figure 3 indeed shows that these eigenvalues of F tend to fill out $\mathcal{B}(F)$.

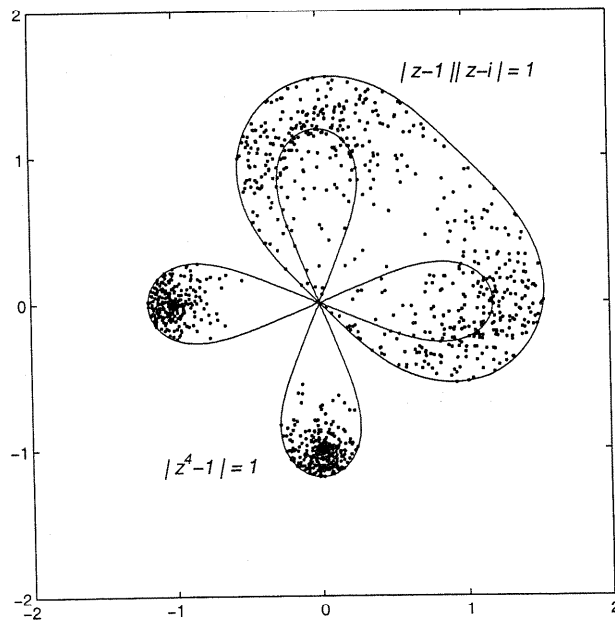


Figure 3

It is of interest to note the following **near paradox** arising from Theorem 2. As an example, F of (6.1) is irreducible, and it is a known result of Brualdi [3, Cor. 2.11] that a boundary point z of $\mathcal{B}(F)$ can be an eigenvalue of F only if z is a boundary point of *each* of the lemniscates $\mathcal{B}_{\gamma_1}(F)$ and $\mathcal{B}_{\gamma_2}(F)$ of (6.2). But from Figure 2, it is apparent that $z = 0$ is the *only* point for which $\partial\mathcal{B}_{\gamma_1}(F)$ and $\partial\mathcal{B}_{\gamma_2}(F)$ have a common point. Yet, (4.14) of Theorem 2 gives the nearly contradictory result that, for *each* point of $\partial\mathcal{B}(F)$, there is an arbitrarily close eigenvalue of some matrix in $\Omega_{\mathcal{B}}(F)$. The difference, of course, lies in the fact that Cor. 2.11 of [2] applies to the *fixed* matrix F , while the common data of (4.2) apply to *all* matrices in $\Omega_{\mathcal{B}}(F)$.

Next, we consider the minimal Geršgorin set, $\mathcal{G}^{\mathcal{R}}(F)$, for the matrix F of (6.1). The boundary of this set, $\partial\mathcal{G}^{\mathcal{R}}(F)$, can be verified to be

$$(6.4) \quad \partial\mathcal{G}^{\mathcal{R}}(F) = \left\{ z \in \mathbb{C} : |z^4 - 1| = \frac{1}{2}(|z + 1| \cdot |z + i|) + \frac{1}{2} \right\},$$

and $\partial\mathcal{G}^{\mathcal{R}}(F)$ is shown in Figure 4. Note that $\partial\mathcal{G}^{\mathcal{R}}(F)$ consists of **three** separate components.

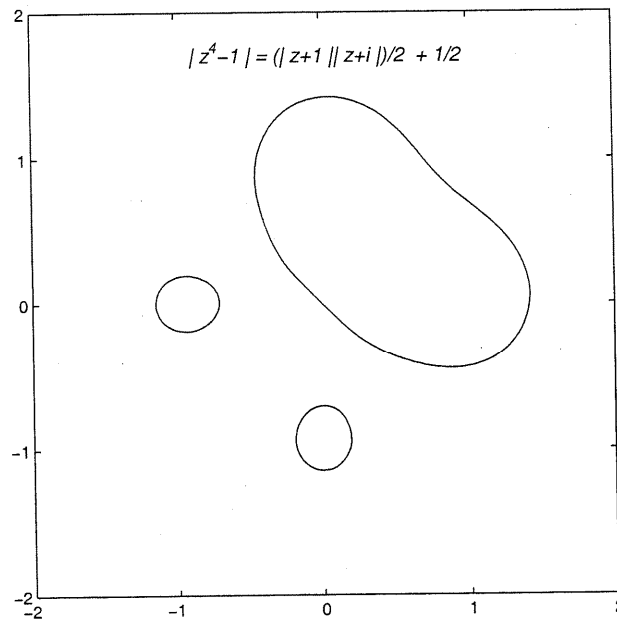


Figure 4

Next, we see from (5.4) that any matrix J in $\Delta(F)$ is of the form

$$(6.5) \quad J = \begin{bmatrix} 1 & \tau_1 e^{i\theta_1} & 0 & 0 \\ \frac{\tau_2 e^{i\theta_2}}{2} & i & \frac{\tau_3 e^{i\theta_3}}{2} & 0 \\ 0 & 0 & -1 & \tau_4 e^{i\theta_4} \\ \tau_5 e^{i\theta_5} & 0 & 0 & -i \end{bmatrix},$$

where $0 \leq \tau_i \leq 1$ and $0 \leq \theta_i \leq 2\pi$, for all $1 \leq i \leq 5$. We similarly consider the eigenvalues of J of (6.5), where we take random choices of $\{s_i\}_{i=1}^5$ in $[0, 1]$, and random choices of

$\{\theta_i\}_{i=1}^5$ in $[0, 2\pi]$ in (6.5). These eigenvalues are plotted in Figure 5, which show again how they fill out $\mathcal{G}^{\mathcal{R}}(F)$. In Figure 6, we show both $\mathcal{B}(F)$ and $\mathcal{G}^{\mathcal{R}}(F)$, and we directly see that

$$(6.6) \quad \mathcal{G}^{\mathcal{R}}(F) \subsetneq \mathcal{B}(F).$$

That (6.6) holds can be seen from the fact that each matrix J of (6.5) is necessarily a matrix H of (6.4), but not conversely.

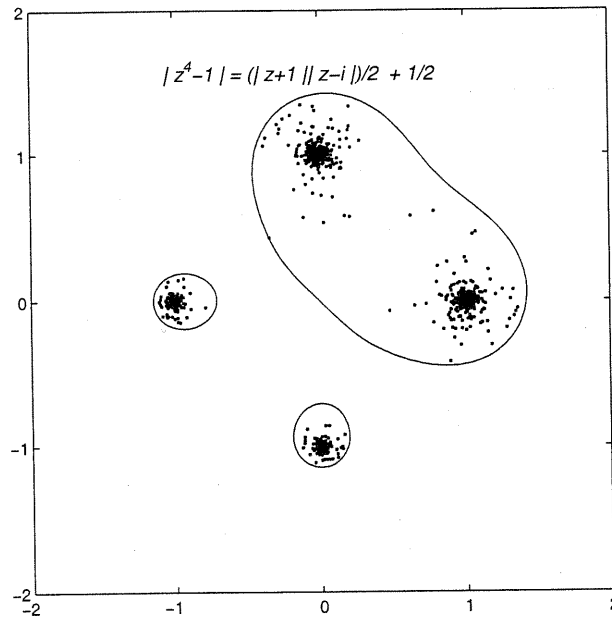


Figure 5

7. A Final Equality. The result, of (5.9) of Theorem 4, shows that the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ is always a subset of the Braudli lemniscate set. Adding to this from the inclusion of (3.1) and (1.6), we then have

$$(7.1) \quad \mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A) \subseteq K(A) \subseteq \mathcal{G}(A),$$

for any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, which is weakly irreducible. But, the last three sets in (7.1) have *no* dependence on weighted row sums, while the first set certainly does from its definition in (5.3). To see the effect that weighted sums can have, consider any $\mathbf{x} = [x_1, x_2, \dots, x_n] > \mathbf{0}$, and with $r_i^{\mathbf{x}}(A)$ of (5.1), we define (cf. (1.3)), in analogy with (5.2) and (5.3),

$$(7.2) \quad K_{i,j}^{\mathbf{x}}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i^{\mathbf{x}}(A) \cdot r_j^{\mathbf{x}}(A)\} \quad (1 \leq i, j \leq n; i \neq j),$$

and (cf. (1.4))

$$(7.3) \quad \mathcal{K}^{\mathbf{x}}(A) := \bigcup_{\substack{i,j=1 \\ i \neq j}}^n K_{i,j}^{\mathbf{x}}(A), \text{ and } \mathcal{K}^{\mathcal{R}}(A) := \bigcap_{\mathbf{x} > \mathbf{0}} \mathcal{K}^{\mathbf{x}}(A).$$

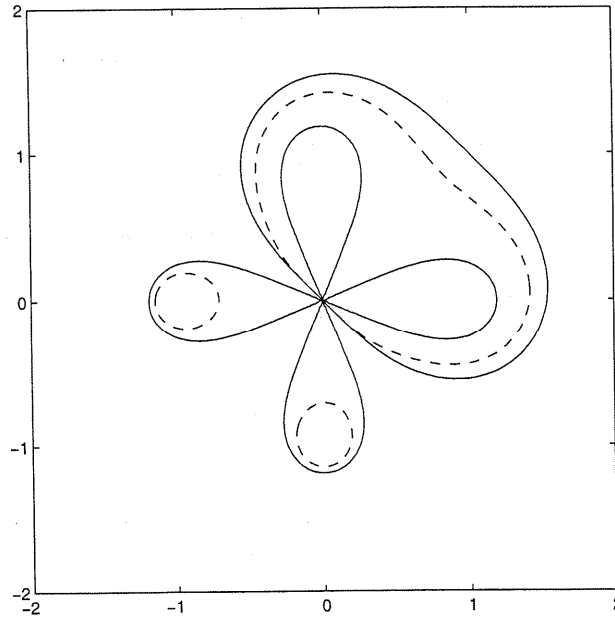


Figure 6

Similarly, for any circuit γ of $\Gamma(A)$, we similarly define (cf. (3.4) and (2.7))

$$(7.4) \quad \mathcal{B}_\gamma^x(A) := \{z \in \mathbb{C} : \prod_{i \in \gamma} |z - a_{i,i}| \leq \prod_{i \in \gamma} r_i^x(A)\},$$

and

$$(7.5) \quad \mathcal{B}^x(A) = \bigcup_{\gamma \in \mathcal{C}(A)} \mathcal{B}_\gamma^x(A), \text{ and } \mathcal{B}^{\mathcal{R}}(A) := \bigcap_{x > 0} \mathcal{B}^x(A).$$

As we see from (1.1) and (5.2), $r_i^x(A) = r_i(X^{-1}AX)$. Hence, the last three inclusions of (7.1), applied to the matrix $X^{-1}AX$, become

$$(7.6) \quad \mathcal{B}^x(A) \subseteq \mathcal{K}^x(A) \subseteq \mathcal{G}^x(A),$$

and as (7.6) holds for any $x > 0$, then

$$(7.7) \quad \mathcal{B}^{\mathcal{R}}(A) \subseteq \mathcal{K}^{\mathcal{R}}(A) \subseteq \mathcal{G}^{\mathcal{R}}(A).$$

On the other hand, we have from (5.9) and (7.4) that

$$\mathcal{G}^{\mathcal{R}}(X^{-1}AX) \subseteq \mathcal{B}(X^{-1}AX) = \mathcal{B}^x(A), \text{ for any } x > 0.$$

But as is easily verified, $\mathcal{G}^{\mathcal{R}}(X^{-1}AX) = \mathcal{G}^{\mathcal{R}}(A)$ for any $x > 0$, so that

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}^x(A) \text{ for any } x > 0.$$

As this inclusion holds for all $x > 0$, then with (7.5),

$$(7.8) \quad \mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}^{\mathcal{R}}(A).$$

Thus, on combining (7.7) and (7.8), we immediately have the result of

Theorem 5. *For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, which is weakly irreducible, then*

$$(7.9) \quad \mathcal{G}^{\mathcal{R}}(A) = \mathcal{K}^{\mathcal{R}}(A) = \mathcal{B}^{\mathcal{R}}(A).$$

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