Alternating Direction Iteration Methods For 
$n$ Space Variables

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We consider the iterative solution of the system of linear equations

\[(X_1 + X_2 + \cdots + X_n)z = f, \quad n \geq 2,\]

where each \(X_j, 1 \leq j \leq n,\) is a Hermitian and positive definite \(N \times N\) matrix. If \(n = 2,\) the iterative methods of Peaceman-Rachford [1, Chapter 7], or D'yakonov [2] and Kellogg [3], may be used to solve (1). In this paper these methods are generalized to \(n \geq 2,\) and are shown, in a sense, to be dual to one another.

Let \(\rho > 0\) be fixed, and define \(z_j = (\rho I + X_j)z.\) From (1) we get the compound \(nN \times nN\) matrix equation

\[
\begin{bmatrix}
I & -W_1(\rho) & \cdots & -W_n(\rho) \\
-W_1(\rho) & I & \cdots & -W_n(\rho) \\
\vdots & \vdots & \ddots & \vdots \\
-W_1(\rho) & -W_2(\rho) & \cdots & I
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_n
\end{bmatrix}
=
\begin{bmatrix}
f_1 \\
f_2 \\
\vdots \\
f_n
\end{bmatrix},
\]

where

\[(3) \quad W_j(\rho) = (\rho I + X_j)^{-1} \left( \frac{\rho}{n-1} I - X_j \right).
\]

Our first set of alternating direction iterative methods will be the block Jacobi and block Gauss-Seidel iterative methods applied to (2), namely

\[(4) \quad (\rho I + X_j)u_j^{(n+1)} = \sum_{k \neq j} \left( \frac{\rho}{n-1} I - X_k \right) u_k^{(n)} + f, \quad 1 \leq j \leq n,\]

and

\[
(5) \quad (\rho I + X_j)u_j^{(n+1)} = \sum_{k \neq j} \left( \frac{\rho}{n-1} I - X_k \right) u_k^{(n+1)} + \sum_{k \neq j} \left( \frac{\rho}{n-1} I - X_k \right) u_k^{(n)} + f.
\]

If \(n = 2,\) (5) is the Peaceman-Rachford method.

We now form the transpose of the matrix of (2), and consider the compound matrix equation

\[
\begin{bmatrix}
I & -W_1(\rho) & \cdots & -W_n(\rho) \\
-W_1(\rho) & I & \cdots & -W_n(\rho) \\
\vdots & \vdots & \ddots & \vdots \\
-W_1(\rho) & -W_2(\rho) & \cdots & I
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
=
\begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n
\end{bmatrix}.
\]

If the block Jacobi and block Gauss-Seidel iterative methods are applied to (6),

Received February 4, 1963.
one obtains the alternating direction iterative methods

\begin{equation}
(\rho I + X_j) y_j^{(m+1)} = \left( \frac{\rho}{n-1} I - X_j \right) \sum_{k \neq j} y_k^{(m)} + f_j, \quad 1 \leq j \leq n,
\end{equation}

and

\begin{equation}
(\rho I + X_j) y_j^{(m+1)} = \left( \frac{\rho}{n-1} I - X_j \right) \left\{ \sum_{k < j} y_k^{(m+1)} + \sum_{k > j} y_k^{(m)} \right\} + f_j, \quad 1 \leq j \leq n.
\end{equation}

Here \( f_j = (\rho I + X_j) g_j \), and it is assumed that

\begin{equation}
f_1 + \cdots + f_n = f.
\end{equation}

When \( n = 2 \), (8) is the method of D'yakonov. Thus, the Peaceman-Rachford iterative method and D'yakonov's method (and their generalization) are dual to one another in the sense that either can be viewed as the Gauss-Seidel iterative method applied to a particular composite matrix or its transpose.

Since each matrix \( X_j \) is Hermitian and positive definite, let the eigenvalues \( \lambda_i(j) \) of \( X_j \) satisfy

\[ 0 < a \leq \lambda_i(j) \leq b, \quad 1 \leq i \leq N, \quad 1 \leq j \leq n. \]

**Theorem.** If \( \rho > (n - 2) b / 2 \), and \( \{ u_j^{(m)} \} \) is defined by (4) or (5), and \( \{ y_j^{(m)} \} \) is defined by (7) or (8), then

\begin{equation}
\lim_{m \to \infty} u_j^{(m)} = z \quad \text{for each} \quad 1 \leq j \leq n,
\end{equation}

and

\begin{equation}
\lim_{m \to \infty} (y_1^{(m)} + \cdots + y_n^{(m)}) = z,
\end{equation}

where \( z \) is the solution of \( (1) \).

**Proof.** Using spectral (\( L_2 \)) norms, it is easy to see that there exists a \( q < 1 \) such that

\[ \| W_j(\rho) \| = \max_{1 \leq i \leq N} \left| \frac{\rho}{n-1} - \frac{\rho - \lambda_i(j)}{\rho + \lambda_i(j)} \right| \leq \frac{q}{n-1} < \frac{1}{n-1}, \quad 1 \leq j \leq n \]

for \( \rho > \left( \frac{n - 2}{2} \right) b \). Letting

\[ T = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \]

denote a column vector with \( nN \) components it is readily verified that the quantity

\begin{equation}
\| T \| = \max_{1 \leq i \leq n} \| v_i \|
\end{equation}

satisfies all the axioms for a vector norm, where in particular we are using Euclidean norms for the subvectors \( v_i \) of \( T \). Let us denote the compound matrix of (2) by
$I - W$. It follows that for all $\xi$ and all $\rho > \left(\frac{n - 2}{2}\right)b$,

$$
\| W\xi \| \leq q \| \xi \| < \| \xi \|
$$

so $W$ is a convergent matrix. But, as $W$ is just the block Jacobi iteration matrix derived from (2), the block Jacobi iterative method of (4) is convergent. If $W^*$ is the conjugate transpose of $W$, the same argument shows that $\| W^*\xi \| < \| \xi \|$; so the iterative method of (7) is also convergent. A similar argument shows that the block Gauss-Seidel methods (5) and (8) are convergent.

If $u_j = \lim_{n \to n} u_j^{(n)}$ in (4) or (5), the $x_j$ satisfy the system of equations

$$(\rho I + X_j)u_j = \sum_{k \neq j} \left(\frac{\rho}{n - 1} I - X_k\right)u_k + f, \quad 1 \leq j \leq n.$$  

Using (13), it may be seen that this system has a unique solution. Since $u_j = z$, $1 \leq j \leq n$, is a solution, (10) is obtained.

If $y_j = \lim_{n \to n} y_j^{(n)}$ in (7) or (8), the $y_j$ satisfy the system of equations

$$(\rho I + X_j)y_j = \left(\frac{\rho}{n - 1} I - X_j\right)\sum_{k \neq j} y_k + f, \quad 1 \leq j \leq n.$$  

Adding these, one obtains $(X_1 + \cdots + X_n)(y_1 + \cdots + y_n) = f$, so that (11) is obtained, proving the theorem.

We remark that this theorem can also be deduced as an application of a generalization [4] of the well known result of Collatz [5], viz., that a strictly diagonally dominant matrix gives rise to convergent Jacobi and Gauss-Seidel iterative methods. For the norms of (12), the partitioned matrix of (2) or (6) is block strictly diagonally dominant in the sense of [6].

Because of the restriction $\rho > (n - 2)b/2$, it is doubtful that this procedure converges very rapidly, and for this reason, no estimates of rates of convergence are included. (This restriction on $\rho$ is necessary even in the favorable case when the $X_j$ all commute with one another.) We stress, however, that the main point of this paper is the theoretical result of convergence without commutativity assumptions on the matrices $X_j$. To our knowledge, similar results have not been proved for other alternating direction methods applied to $n$-dimensional problems, $n \geq 3$. Complementary to this is the fact that three-dimensional matrix problems have been constructed* for which the Douglas-Rachford method [7] and the generalized Peaceman-Rachford method of Douglas [8] each diverge for a suitable single positive parameter $\rho$.

Finally, it is worth mentioning that our generalization of the Peaceman-Rachford iterative method (5) is computationally more attractive than our generalization of the method of D'yakonov, since the latter requires, from (11), more vector storage in practical applications.

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