APPROXIMATION OF THE MINIMAL GERŠGORIN SET
OF A SQUARE COMPLEX MATRIX* 

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Abstract. In this paper, we address the problem of finding a numerical approximation to the minimal Geršgorin set, \(\Gamma^R(A)\), of an irreducible matrix \(A\) in \(\mathbb{C}^{n \times n}\). In particular, boundary points of \(\Gamma^R(A)\) are related to a well-known result of Olga Taussky.

Key words. eigenvalue localization, Geršgorin theorem, minimal Geršgorin set.

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1. Introduction. Given an irreducible matrix \(A = [a_{i,j}]\) in \(\mathbb{C}^{n \times n}\), its \(i\)-th Geršgorin disk is defined, with \(N := \{1, 2, \ldots, n\}\), by

\[
\Gamma_i(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{i,j}|\} \quad (i \in N),
\]

and the union of all these disks, denoted by

\[
\Gamma(A) := \bigcup_{i \in N} \Gamma_i(A),
\]

is called the Geršgorin set for \(A\). A well-known result of Geršgorin [2] gives us that \(\Gamma(A)\) contains the spectrum, \(\sigma(A)\), of \(A\), i.e.,

\[
\sigma(A) := \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\} \subseteq \Gamma(A).
\]

Continuing, for any \(x = [x_1, x_2, \ldots, x_n]^T > 0\) in \(\mathbb{R}^n\), i.e., \(x_i > 0\) for all \(i \in N\), let \(X := \text{diag}[x_1, x_2, \ldots, x_n]\) denote the associated nonsingular diagonal matrix. Then, \(X^{-1}AX\) has the same eigenvalues as \(A\). Thus, with the Geršgorin disks for \(X^{-1}AX\) now given by

\[
\Gamma_i^X(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i^X(A) := \sum_{j \in N \setminus \{i\}} \frac{|a_{i,j}|x_j}{x_i}\} \quad (i \in N),
\]

and with the associated Geršgorin set,

\[
\Gamma^X(A) := \bigcup_{i \in N} \Gamma_i^X(A),
\]

then

\[
\sigma(A) \subseteq \Gamma^X(A), \quad \text{for any} \ x > 0 \ \text{in} \ \mathbb{R}^n.
\]

The inclusion of (1.6) is also a well-known result of Geršgorin [2]. Clearly, the following intersection,

\[
\Gamma^R(A) := \bigcap_{x > 0 \ \text{in} \ \mathbb{R}^n} \Gamma^X(A),
\]

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called the *minimal Gershgorin set* in [4, 6], is always a subset of \( \Gamma^* (A) \), for any \( x > 0 \) in \( \mathbb{R}^n \), thereby giving the sharpest inclusion set for \( \sigma (A) \), with respect to all positive diagonal similarity transforms \( X^{-1} A X \) of \( A \).

This sharpness can also be expressed in the following way; cf. [6, Theorem 4.5]. With

\[
\hat{\Omega}(A) := \{ B = [b_{i,j}] \in \mathbb{C}^{n \times n} : b_{i,i} = a_{i,i} \text{ and } |b_{i,j}| \leq |a_{i,j}| \text{ for } i \neq j \ (i, j \in N) \},
\]

then

\[
\sigma (\hat{\Omega}(A)) := \bigcup_{B \in \hat{\Omega}(A)} \sigma (B) = \Gamma^R (A),
\]

i.e., each point of \( \Gamma^R (A) \) is an eigenvalue of *some* matrix \( B \) in \( \hat{\Omega}(A) \).

Unlike the Gershgorin set \( \Gamma (A) \) of (1.2) or \( \Gamma^* (A) \) of (1.5), the minimal Gershgorin set \( \hat{\Omega}(A) \) of (1.7) is not in general easy to determine numerically. The aim of this paper is to find a *reasonable approximation* of \( \Gamma^R (A) \), with a finite number of calculations, which contains \( \Gamma^R (A) \), and for which a limited number of boundary points of this approximation are actual boundary points of \( \Gamma^R (A) \). The determination of these latter boundary points are then related to a famous sharpening, by Olga Taussky [3], of the Gershgorin set of (1.2).

2. Background. Given an irreducible matrix \( A = [a_{i,j}] \in \mathbb{C}^{n \times n} \), its associated irreducible matrix \( Q(z) = [q_{i,j}(z)] \), in \( \mathbb{R}^{n \times n} \), is defined by

\[
q_{i,j}(z) := |z - a_{i,i}|, \text{ and } q_{i,j}(z) := |a_{i,j}|, \text{ for } i \neq j \ (i, j \in N).
\]

If

\[
\mu(z) := \max_{i \in N} |z - a_{i,i}|
\]

then the matrix \( B(z) := [B_{i,j}(z)] \in \mathbb{R}^{n \times n} \), defined by

\[
B_{i,i}(z) := \mu(z) - |z - a_{i,i}|, \text{ and } B_{i,j}(z) := |a_{i,j}|, \ i \neq j \ (i, j \in N),
\]

satisfies

\[
B(z) = Q(z) + \mu(z) I_n,
\]

where \( B(z) \) is a nonnegative irreducible matrix in \( \mathbb{R}^{n \times n} \). Then, from the Perron-Frobenius theory of nonnegative matrices, the matrix \( B(z) \) possesses a positive real eigenvalue, \( \rho(B(z)) \), called the *Perron root* of \( B(z) \), which is characterized as follows. For any \( x > 0 \) in \( \mathbb{R}^{n \times n} \), either

\[
\min_{i \in N} \{ (B(z)x)_i / x_i \} < \rho(B(z)) < \max_{i \in N} \{ (B(z)x)_i / x_i \},
\]

or

\[
B(z)x = \rho(B(z))x.
\]

Thus, if we set

\[
\nu(z) := \rho(B(z)) - \mu(z) \ (\text{all } z \in \mathbb{C}),
\]

then \( \nu(z) \) is a real-valued function, defined for all \( z \in \mathbb{C} \). Moreover, from (2.5) and (2.6), for any \( x > 0 \) in \( \mathbb{R}^{n} \) and any \( z \in \mathbb{C} \), either

\[
\min_{i \in N} \{ (Q(z)x)_i / x_i \} < \nu(z) < \max_{i \in N} \{ (Q(z)x)_i / x_i \},
\]
or

\[(2.9) \quad Q(z)x = \nu(z)x,\]

the last equation giving us that \(\nu(z)\) is an eigenvalue of \(Q(z)\).

The following connection of the function \(\nu(z)\) of (2.7) to the minimal Geršgorin set, \(\Gamma^R(A)\), comes from [4] and [6]:

\[(2.10) \quad z \in \Gamma^R(A) \text{ if and only if } \nu(z) \geq 0,\]

and

\[(2.11) \quad \text{if } z \in \partial \Gamma^R(A), \text{ then } \nu(z) = 0.\]

It is also known (cf. [6], Theorem 4.6), from the assumption that \(A\) is irreducible, that

\[(2.12) \quad \nu(a_{i,i}) > 0, \text{ for all } i \in N.\]

Further, given any real number \(\theta\) with \(0 \leq \theta < 2\pi\), it is known (cf. [6], Theorem 4.6) that there is a largest number \(\hat{\theta}(\theta) > 0\) such that

\[(2.13) \quad \nu(a_{i,i} + \hat{\theta}(\theta)e^{i\theta}) = 0, \quad \nu(a_{i,i} + te^{i\theta}) \geq 0, \text{ for all } 0 \leq t \leq \hat{\theta}(\theta),\]

so that the entire complex interval \([a_{i,i} + te^{i\theta}],_{t=0}^{2\pi}\hat{\theta}(\theta)\) lies in \(\Gamma^R(A)\). This implies that the set

\[(2.14) \quad \bigcup_{\theta=0}^{2\pi} [a_{i,i} + te^{i\theta}],_{t=0}^{\hat{\theta}(\theta)}\]

is a star-shaped subset of \(\Gamma^R(A)\), for each \(i \in N\), with

\[(2.15) \quad \nu(a_{i,i} + \hat{\theta}(\theta)e^{i\theta}) \in \partial \Gamma^R(A).\]

The results of (2.14) and (2.15) will be used below.

Next, we recall the famous result of Olgla Taussky [3], on a sharpening of the Geršgorin Circle Theorem: Let \(A = [a_{i,j}]\) in \(\mathbb{C}^{n \times n}\) be irreducible. If \(\lambda \in \sigma(A)\) is such that \(\lambda \notin \text{int } \Gamma_i(A)\) for each \(i \in N\), i.e., \(|\lambda - a_{i,i}| > r_i(A)\) for each \(i \in N\), then

\[(2.16) \quad |\lambda - a_{i,i}| = r_i(A), \text{ for each } i \in N,\]

i.e., each Geršgorin circle \(\{z \in \mathbb{C} : |z - a_{i,i}| = r_i(A)\}\) passes through \(\lambda\).

To complete this section, we include the following:

\[(2.17) \quad \text{If } \nu(z) = 0, \text{ then } \det Q(z) = 0.\]

This follows directly from (2.9), since \(\nu(z)\) is an eigenvalue of \(Q(z)\). Finally, from [6, Exercise 7, p. 108], we also have that

\[(2.18) \quad \text{for any } z \text{ and } z' \in \mathbb{C}, |\nu(z) - \nu(z')| \leq |z - z'|,\]

so that \(\nu(z)\) is uniformly continuous in \(\mathbb{C}\). This also will be used below.
3. Numerical procedure for approximating $\Gamma^R(A)$. With the given irreducible matrix $A = [a_{ij}]$ in $\mathbb{C}^{n \times n}$, choose any $j$ in $N$, and set $z = a_{ij}$. Next, we assume that the nonnegative irreducible matrix $B(a_{ij})$, which has at least one zero diagonal entry from (2.3), is a primitive matrix; cf. of [5, Section 2.2]. (We note that this is certainly the case if some diagonal entry of $B(a_{ij})$ is positive. More generally, if $B(a_{ij})$ is not primitive (i.e., $B(a_{ij})$ is cyclic of some index $p \geq 2$), then any simple shift of $B(a_{ij})$ into $B(a_{ij}) + \varepsilon I_n$ is primitive for each $\varepsilon > 0$.)

With $B(a_{ij})$ primitive, then, starting with an $x^{(0)} > 0$ in $\mathbb{R}^n$, the power method gives convergent upper and lower estimates for $\rho(B(a_{ij}))$, i.e., if $x^{(m)} := B^m(a_{ij})x^{(0)}$ for all $m \geq 1$, then with $x^{(m)} := [x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}]^T$, we have

$$(3.1) \quad \lambda_m := \min_{i \in N} \left\{ \frac{x_i^{(m+1)}}{x_i^{(m)}} \right\} \leq \rho(B(a_{ij})) \leq \max_{i \in N} \left\{ \frac{x_i^{(m+1)}}{x_i^{(m)}} \right\} =: \overline{\lambda}_m$$

for all $m \geq 1$, with

$$(3.2) \quad \lim_{m \to \infty} \lambda_m = \rho(B(a_{ij})) = \lim_{m \to \infty} \overline{\lambda}_m.$$

In this way, using (2.4), (2.7), and (2.9), convergent upper and lower estimates of $\nu(a_{ij})$ can be numerically obtained. (These estimations of $\nu(a_{ij})$ do not need great accuracy for graphing purposes, as the example in Section 4 shows.)

Next, assume, for convenience, that $\nu(a_{ij}) > 0$ is accurately known, and select any real $\theta$, with $0 \leq \theta < 2\pi$. The numerical goal now is to estimate the largest $\hat{\theta}_j(\theta)$, with sufficient accuracy, where, from (2.2),

$$(3.3) \quad \nu(a_{ij} + \hat{\theta}_j(\theta)e^{i\theta}) = 0, \quad \nu(a_{ij} + (\hat{\theta}_j(\theta) + \varepsilon)e^{i\theta}) < 0$$

for all sufficiently small $\varepsilon > 0$. By definition, we then have that

$$(3.4) \quad a_{ij} + \hat{\theta}_j(\theta)e^{i\theta} \text{ is a boundary point of } \Gamma^R(A).$$

This means, from the min-max conditions (2.8)-(2.9), that there is an $x > 0$, in $\mathbb{R}^n$, such that (cf. (2.9))

$$(3.5) \quad Q(a_{ij} + \hat{\theta}_j(\theta)e^{i\theta})x = 0, \text{ where } x = [x_1, x_2, \ldots, x_n]^T > 0.$$

Equivalently, on calling $a_{ij} + \hat{\theta}_j(\theta)e^{i\theta} =: z_j(\theta)$, we can express (3.5), using the definition of (2.1), as

$$(3.6) \quad |z_j(\theta) - a_{ij}| = \sum_{k \in N \setminus \{i\}} |a_{i,k}x_k/x_{i,j}|, \text{ (all } i \in N),$$

which can be interpreted, from (2.16), simply as Olga Taussky's boundary result. What is perhaps more interesting is that it is geometrically unnecessary now to determine the components of the vector $x > 0$ in $\mathbb{R}^n$, for which (3.6) is valid. This follows since knowing the boundary point $z_j(\theta)$ of $\Gamma^R(A)$, and knowing of the centers, $\{a_{i,j}\}_{i \in N}$, of the associated $n$ Geršgorin disks, then all the circles of (3.6) can be directly drawn, without knowing the components of the vector $x$. 
We return to the numerical estimation of \( \hat{\gamma}_j(\theta) \), which satisfies (3.3)-(3.5). Setting \( z := a_{j, j} \) and \( z' := a_{j, j} + \hat{\gamma}_j(\theta)e^{i\theta} \), we know from (2.18) that

\[
\hat{\gamma}_j(\theta) \geq \nu(a_{j, j}) > 0.
\]

Consider then the number \( \nu(a_{j, j} + \nu(a_{j, j})e^{i\theta}) \). If this number is positive, then increase the number \( \nu(a_{j, j}) \) to \( \nu(a_{j, j}) + \Delta, \Delta > 0 \), until \( \nu(a_{j, j} + (\nu(a_{j, j}) + \Delta)e^{i\theta}) \) is negative, and apply a bisection search to the interval \( [\nu(a_{j, j}), \nu(a_{j, j}) + \Delta] \) to determine \( \hat{\gamma}_j(\theta) \), satisfying (3.3). (Again, as in the estimation of \( \nu(a_{j, j}) \), estimates of \( \hat{\gamma}_j(\theta) \) do not need great accuracy for graphing purposes.) We remark that a similar bisection search, on \( z \), can be directly applied to

\[
\det Q(\nu(a_{j, j} + \hat{\gamma}_j(\theta)e^{i\theta})) = 0,
\]

as a consequence of (2.11) and (2.15), but this requires, however, the evaluation of an \( n \times n \) determinant.

To summarize, given an irreducible matrix \( A = [a_{i, j}] \) in \( \mathbb{C}^{n \times n} \), our procedure for approximating its minimal Geršgorin set, \( \Gamma^R(A) \), is to first determine, with reasonable accuracy, the positive numbers \( \{\nu(a_{j, j})\}_{j \in N} \), and then, again with reasonable accuracy, to determine a few boundary points \( \{\omega_k\}_{k=1}^m \) of \( \Gamma^R(A) \). For each such boundary point \( \omega_k \) of \( \Gamma^R(A) \), there is an associated Geršgorin set, consisting of the union of the \( n \) Geršgorin disks, namely,

\[
\Gamma^{\omega_k}(A) := \bigcup_{i \in N} \{z \in \mathbb{C} : |z - a_{i, i}| \leq |\omega_k - a_{i, i}|\},
\]

and their intersection,

\[
\bigcap_{k=1}^m \Gamma^{\omega_k}(A),
\]

gives an approximation to \( \Gamma^R(A) \), for which \( \Gamma^R(A) \) is a subset, and for which \( m \) points, of the boundary of \( \bigcap_{k=1}^m \Gamma^{\omega_k}(A) \), are boundary points of \( \Gamma^R(A) \).

4. An easy example. Consider the irreducible \( 3 \times 3 \) matrix

\[
C = \begin{bmatrix}
2 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix},
\]

whose minimal Geršgorin set, \( \Gamma^R(C) \), is shown with the inner blue boundary in Figure 4.1. (This minimal Geršgorin set, \( \Gamma^R(C) \), also appears as the set with boundary \((1) (2) (3)\) of \([6, Figure 4.4] \).) For the vector \( x_0 = [1, 1, 1]^T \in \mathbb{R}^3 \), the associated Geršgorin set \( \Gamma^{x_0}(C) \), turns out to be simply

\[
\Gamma^{x_0}(C) = \{z \in \mathbb{C} : |z - 2| \leq 2\}.
\]

The boundary of this set is the (outer) black circle in Figure 4.1.

Next, starting with the diagonal entry, \( z = 2 \), of the matrix \( C \), we estimate \( \nu(2) \), which is positive from (2.12). As \( \mu(2) = 1 \) from (2.2), the associated nonnegative irreducible matrix \( B(2) \) from (2.3) is

\[
B(2) = \begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]
and a few power method iterations (see (3.1)-(3.2)), starting with \( x_0 = [2, 1, 2]^T \), gives that 
\[ \rho(B(2)) = 2.2. \] More precisely, \( \rho(B(2)) = 2.24697 \), so that from (2.7) we have \( \nu(2) = 1.24697 \). 

Next, we search on the ray \( 2 + t \), with \( t \geq 0 \), for the largest value \( \tilde{t} \) for which \( \nu(2 + \tilde{t}) = 0 \) and \( \nu(2 + t) \geq 0 \) for all \( 0 \leq t \leq \tilde{t} \). Using the inequality of (2.18), it follows that 
\[ \tilde{t} \geq \nu(2) = 1.24697. \] However, in this particular case, it happens that \( \tilde{t} = 1.24697 \), so that 
\[ z_1 = 3.24697 \] is such that \( \nu(z_1) = 0 \), with \( z_1 \in \partial \Gamma^R(C) \). Similarly, on considering the diagonal entry \( 1 = c_{2,2} \), we approximate \( \nu(1) \), which turns out to be \( \nu(1) = 0.80194 \), and then searching on the ray \( 1 - t \), \( t \geq 0 \), we similarly obtain \( \nu(z_2) = 0 \) with \( z_2 = 0.19806 \), and with \( z_2 \in \partial \Gamma^R(C) \). Calling \( \Gamma^R(C) \) and \( \Gamma^R(C) \) the associated Geršgorin sets, then the intersection of the three sets, \( \bigcap_{U=0}^3 \Gamma^R(C) \), is shown in Figure 4.1 with the red boundary, where the boundary of the minimal Geršgorin set, \( \Gamma^R(C) \), is shown in blue.

We see from Figure 4.1 that the set with the red boundary is a set in the complex plane which contains \( \Gamma^R(C) \) and has two real boundary points, shown as the black squares \( z_1 \) and \( z_2 \), in common with \( \Gamma^R(C) \). Continuing, knowing \( \nu(a_{1,1} = a_{3,3} = 2) = 1.24697 \) and 
\[ \nu(a_{2,2} = 1) = 0.80194, \] we then look for four additional points of \( \partial \Gamma^R(C) \) which are found on the four rays: \( 2 \pm it, t \geq 0 \), and \( 1 \pm it, t \geq 0 \). This gives us the following four points 
\[ \{z_j\}_{j=3}^6 \] of \( \Gamma^R(C) \): 
\[ z_3 = 1 + i(1.150963), \] 
\[ z_4 = 2, \] 
\[ z_5 = 2 + i(1.34236), \] 
\[ z_6 = \overline{2}. \]

The intersection now of the above associated six Geršgorin sets is shown in Figure 4.1 with the green boundary, which includes \( \Gamma^R(C) \) and has six boundary points in common with \( \partial \Gamma^R(C) \), shown as solid black squares. The region between the green boundary of \( \Gamma^R(C) \) and its blue boundary is colored in yellow, which can be seen as small "roofs" composed of segments of circular arcs.

The amount of numerical calculation to obtain a good approximation to \( \Gamma^R(C) \) is moderate. It is further evident that better approximations to \( \Gamma^R(C) \), having more points in common with \( \partial \Gamma^R(C) \), can be similarly constructed.

5. Comparisons with Brualdi sets. Given an irreducible matrix \( A = [a_{ij}] \) in \( \mathbb{C}^{n \times n} \), \( n \geq 2 \), one can similarly associate with \( A \) a minimal Brauer set, \( \mathcal{K}^R(A) \), as well as a minimal

\[ ^1 \text{All such numbers are truncated after five decimal digits.} \]
Brualdi set $\mathcal{B}^R(A)$, as described in [6, Section 4.3]. However, it is known (see [6, Theorem 4.15]) that all of these sets are equal, i.e.,

\begin{equation}
\Gamma^R(A) = \mathcal{K}^R(A) = \mathcal{B}^R(A),
\end{equation}

but the approximation of, say, the minimal Brualdi set $\mathcal{B}^R(A)$, would now differ from our approximations of the minimal Geršgorin set, $\Gamma^R(A)$, described earlier in this paper. For matrices having a very large number of nonzero off-diagonal entries, it is unlikely (see [6, Section 2.3]) that a similar numerical approximation of the minimal Brualdi set, $\mathcal{B}^R(A)$, which from (5.1) equals $\Gamma^R(A)$, would be numerically competitive with our numerical approach of Section 3 for approximating $\Gamma^R(A)$. But, in the case of the matrix $C$ of (4.1), there are just two associated Brualdi cycles, $\gamma_1 = (13)$ and $\gamma_2 = (23)$, for this matrix $C$, so that the approximation of $\Gamma^R(C)$, via Brualdi sets, in this case, is easy. In particular, for any $x = [x_1, x_2, x_3]^T > 0$ in $\mathbb{R}^3$, its associated Brualdi lemniscates (cf. [6, eq. (4.78)]) are

\begin{equation}
B_{\gamma_1}^*(C) = \{z \in \mathbb{C} : |z - 2| \leq \frac{x_3}{x_1} \cdot \left(\frac{x_1 + x_2}{x_3}\right) = \frac{x_1 + x_2}{x_1},
\end{equation}

and

\begin{equation}
B_{\gamma_2}^*(C) = \{z \in \mathbb{C} : |z - 1||z - 2| \leq \frac{x_3}{x_2} \cdot \left(\frac{x_1 + x_2}{x_3}\right) = \frac{x_1 + x_2}{x_2},
\end{equation}

so that its associated Brualdi set is (cf. [6, eq. (2.40)])

\begin{equation}
B^*(C) = B_{\gamma_1}^*(C) \bigcup B_{\gamma_2}^*(C).
\end{equation}

Now, knowing that $z_1 = 3.24697$ is a boundary point of $\Gamma^R(C)$, we determine $x_1 > 0$ and $x_2 > 0$ so that $z_1 = 3.24697$ is a boundary point of $B_{\gamma_1}^*(C)$. For this particular point $z_1 = 3.24697$, the associated Brualdi set, consisting of the union of two Brualdi lemniscate sets, is such that the boundary of each Brualdi lemniscate passes through $z_1$. (This is exactly the analog of Olga Taussky Theorem in the Geršgorin case; see [1] and [6, Theorem 2.8].) The union of these two Brualdi lemniscate sets can be verified to reduce to

\begin{equation}
B^{z_1}(C) = \{z \in \mathbb{C} : |z - 1||z - 2| \leq 2.86193\}.
\end{equation}

Similarly, for the point $z_2 = 0.19806$, the associated Brualdi set has its two lemniscate sets passing through $z_2$, and the union of these two Brualdi lemniscate sets can be verified to reduce to the disk

\begin{equation}
B^{z_2}(C) = \{z \in \mathbb{C} : |z - 2| \leq 1.80193\}.
\end{equation}

The boundary of the intersection $B_{\gamma_1}^{z_1}(C) \cap B_{\gamma_2}^{z_2}(C)$ is shown in Figure 5.1 with the green boundary. Also shown in Figure 5.1, with the red boundary, is the related Geršgorin set from Figure 4.1, which also has $z_1$ and $z_2$ as common points with the minimal Geršgorin set $\Gamma^R(C)$.

From Figure 5.1, we see that $B_{\gamma_1}^{z_1}(C) \cap B_{\gamma_2}^{z_2}(C)$ is a proper subset of the related Geršgorin set, where the difference between these sets is shown in yellow. This is not unexpected, as it is known (cf. [6, eq. (4.80)]) that, for any matrix $A$ in $\mathbb{C}^{n \times n}$,

\begin{equation}
B^x(A) \subseteq \Gamma^x(A), \quad \text{for any } x > 0 \text{ in } \mathbb{R}^n.
\end{equation}
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