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# On Smallest Isolated Gerschgorin Disks for Eigenvalues. III

HELEN I. MEDLEY and RICHARD S. VARGA

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## 1. Introduction

If a given irreducible $n \times n$ complex matrix $A$ admits $s$, $1 \leqq s \leqq n$, Gerschgorin disks which are isolated from the remaining disks, then it is well known that the union of these $s$ disks contains $s$ eigenvalues of $A$. It has been shown in [5], [9] and [11] that, for the case where $s=1$, algorithms exist which yield the smallest Gerschgorin disk under positive diagonal similarity transformations which contains this isolated eigenvalue of $A$.

The purpose of this note is to extend this idea to the case where $s$ is greater than one. It is shown, under slightly stronger hypotheses, that the analogue of the first algorithm of [11], carries over to this situation.

## 2. Basic Lemmas

Let $A=(\alpha_{i,j})$ be a fixed irreducible $n \times n$ complex matrix, and assume that $A$ admits $s$, $1 \leqq s < n$, isolated Gerschgorin disks, i.e., if

(1) $$d_{k,j} = |\alpha_{k,k} - \alpha_{j,j}|, \quad 1 \leqq k, j \leqq n,$$

and

(2) $$\Lambda_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |\alpha_{i,j}|, \qquad 1 \leqq i \leqq n,$$

then there exists a set $S$, $S < \{1, 2, \ldots, n\}$, such that the number of elements in $S$ is $s$ and

(3) $$d_{k,j} - \Lambda_k - \Lambda_j \geqq 0 \quad \text{for all} \quad k \in S \quad \text{and all} \quad j \notin S.$$

Without loss of generality, we can assume that the first $s$ Gerschgorin disks are isolated, i.e., $S=\{1, 2, \ldots, s\}$. Letting

(4) $$G_i(A) = \left\{ z \,\middle|\, |z - \alpha_{i,i}| \leqq \sum_{\substack{j=1 \\ j \neq i}}^{n} |\alpha_{i,j}| = \Lambda_i \right\},$$

we have that

(5) $$\left( \bigcup_{i=1}^{s} G_i(A) \right) \cap G_j^0(A) = \emptyset \quad \text{for all } j, \text{ with } \quad s < j \leqq n,$$

where $G_j^0(A)$ denotes the interior of $G_j(A)$.

We wish to translate the matrix $A$ so that the isolated disks are "centered" at the origin[1]. The following functions are introduced to facilitate this translation.

[1] We remark that this is the analog in the special case $s=1$ of assuming that $a_{1,1}=0$ in [9] or [11].

Let $C$ be the set of complex numbers, and for $r \geqq 0$, let

$$(6) \qquad K(z_0, r) = \{z \in C \,|\, |z - z_0| \leqq r\}.$$

Let

$$(7) \quad I(s) = \left\{ w \in C \,|\, \exists r \geqq 0 \text{ such that } \bigcup_{i=1}^{s} G_i(A) \subset K(w, r) \text{ and } \bigcup_{i=s+1}^{n} G_i(A) \subset \overline{K'(w, r)} \right\},$$

where $\overline{K'}$ denotes the closure of the complement of $K$. We can associate with each $w \in I(s)$ a minimum $r$, denoted by $r(w)$. This $r(w)$ is characterized by

$$(8) \qquad \partial K(w, r(w)) \cap \partial \left( \bigcup_{i \in S} G_i(A) \right) \neq \emptyset.$$

With this characterization of $r(w)$ and the definition of $I(s)$ given in (7), it is not difficult to show[2] that $r(w)$ is Lipschitz-continuous on $I(s)$ and that $I(s)$ is a closed subset of the complex plane. Furthermore, if $I(s) \neq \emptyset$, there exists a unique $\gamma \in I(s)$ such that $r(\gamma) = \inf_{w \in I(s)} r(w)$.

In the following, we assume[3] that our given matrix $A$ is such that $I(s) \neq \emptyset$. Let the center of the unique smallest circle containing $\bigcup_{i \in S} G_i(A)$ be $\gamma$ and let $\tilde{r} = \inf_{w \in I(s)} r(w) = r(\gamma)$. Consider $A_1 = (a_{i,j}) = A - \gamma I$. We further assume $|A_1| = (|a_{i,j}|)$ is such that not all the circles $|z - |a_{i,i}|| = \Lambda_i$, $i = 1, 2, \ldots, n$, are tangent to each other at $z = \tilde{r}$. From a result due to TAUSSKY [8], the irreducibility of $|A_1|$ (which follows from the irreducibility of $A$) and the above assumption on the $\partial G_i(|A_1|)$ are sufficient to imply $\tilde{r}$ is not an eigenvalue of $|A_1|$ and, hence, $A_1$ does not have an eigenvalue of modulus $\tilde{r}$. Furthermore, from the isolated nature of the $s$ Gerschgorin disks and this assumption on $\partial G_i(|A_1|)$, we now know that the open disk $|z - \gamma| < \tilde{r}$ contains precisely $s$ eigenvalues of $A$, and that the open disk $|z| < \tilde{r}$ contains precisely $s$ eigenvalues of $A_1$. The goal is to reduce the radius of this disk.

Let

$$A_1 = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ is $s \times s$ and $A_{2,2}$ is $(n-s) \times (n-s)$. As in [9] or [11], we set up a comparison matrix $B$ defined as follows:

$$(9) \qquad B = \begin{bmatrix} |A_{1,1}| & |A_{1,2}| \\ -|A_{2,1}| & B_{2,2} \end{bmatrix} = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix},$$

where $B_{2,2} = (\tilde{b}_{i,j})$, $1 \leqq i, j \leqq n-s$, is defined from $A_{2,2}$ in the following manner. If $A_{2,2} = (e_{i,j})$, $1 \leqq i, j \leqq n-s$, then $\tilde{b}_{i,i} = |e_{i,i}|$ and $\tilde{b}_{i,j} = -|e_{i,j}|$ for $i \neq j$.

From our choice of $\gamma$, it follows that

$$(10) \qquad |a_{j,j}| - |a_{i,i}| - \Lambda_j - \Lambda_i \geqq 0 \quad \text{for all } j \notin S \text{ and all } i \in S,$$

where $a_{k,k}$ are the diagonal entries of $A_1$. The assumption on $\partial G_i(|A_1|)$, $i = 1, 2, \ldots, n$ implies that strict inequality is valid in (10) for some pair $(j, i)$, with $j \notin S$ and $i \in S$.

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[2] Cf. reference 6, Lemmas 2.1–2.4.

[3] We remark that it is possible to have $s$ isolated Gerschgorin disks of $A$, $2 \leqq s < n$, with $I(s)$ empty (cf. reference 6, Fig. 2), so that this is an added assumption.

**Lemma 1.** For all $|\sigma| \leqq \tilde{r}$, $B_{2,2}-\sigma I$ is an $H$-matrix.

*Proof.* From the definition of an $H$-matrix given by OSTROWSKI [7], the proof will be complete if we show that $B_{2,2}-|\sigma| I$ is an $M$-matrix, for $|\sigma| \leqq \tilde{r}$. By definition, $B_{2,2}=(\tilde{b}_{i,j})$, $1 \leqq i, j \leqq n-s$, has the proper sign pattern for an $M$-matrix. Consider the case where $|\sigma| = \tilde{r}$ and $B_{2,2}$ is irreducible. For each $j \notin S$ we have, from (10), that $|a_{j,j}| - |a_{i,i}| - \Lambda_j - \Lambda_i \geqq 0$ for all $i \in S$. Therefore, since

$$\tilde{r} = \max_{i \in S} \{|a_{i,i}| + \Lambda_i\},$$

we have

$$|a_{j,j}| - \tilde{r} - \Lambda_j \geqq 0 \qquad \text{for each} \quad j \notin S.$$

Since

$$\Lambda_j \geqq \sum_{\substack{k=1 \\ k \neq j-s}}^{n-s} (-\tilde{b}_{j-s,k}) \qquad \text{for all} \quad j \notin S,$$

we deduce that $B_{2,2}-\tilde{r}I$ is diagonally dominant. From the irreducibility of $B$ (which again follows from the irreducibility of $A$), we know that $B_{2,1}$ contains at least one entry different from zero. If the nonzero entry of $B_{2,1}$ occurs in the $j^{\text{th}}$ row of $B_{2,1}$, then in the $j^{\text{th}}$ row of $B_{2,2}-\tilde{r}I$, we have that $\tilde{b}_{j,j} - \tilde{r} - \sum_{\substack{i=1 \\ i \neq j}}^{n-s} |\tilde{b}_{j,i}| > 0$, and hence $B_{2,2}-\tilde{r}I$ is irreducibly diagonally dominant. But $B_{2,2}-\tilde{r}I$ irreducibly diagonally dominant implies that $B_{2,2}-\tilde{r}I$ is an $M$-matrix [10, p. 85].

If $B_{2,2}$ is reducible, then the proof that $B_{2,2}-\tilde{r}I$ is an $M$-matrix follows in a similar fashion by consideration of the normal form for reducible matrices. It is clear that if $B_{2,2}-\tilde{r}I$ is an $M$-matrix, then $B_{2,2}-|\sigma| I$ is an $M$-matrix for all $|\sigma| \leqq \tilde{r}$, which completes the proof.

As a consequence of Lemma 1, we have that $B_{2,2}-\sigma I$ is nonsingular for all $|\sigma| \leqq \tilde{r}$ and we can now, in the spirit of FIEDLER and PTÁK [1], express $B-\sigma I$, $|\sigma| \leqq \tilde{r}$, as the following product:

$$(11) \qquad \left[\begin{array}{c|c} B_{1,1}-\sigma I - B_{1,2}(B_{2,2}-\sigma I)^{-1} B_{2,1} & B_{1,2}(B_{2,2}-\sigma I)^{-1} \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline B_{2,1} & B_{2,2}-\sigma I \end{array}\right].$$

From (11), we have that

$$\det(B-\sigma I) = \det\left(B_{1,1}-\sigma I - B_{1,2}(B_{2,2}-\sigma I)^{-1} B_{2,1}\right) \cdot \det(B_{2,2}-\sigma I).$$

Therefore, since $\det(B_{2,2}-\sigma I) \neq 0$ for all $|\sigma| \leqq \tilde{r}$, we have that $\det(B-\sigma I)=0$ for some $\sigma$, $|\sigma| \leqq \tilde{r}$, implies that $\det(B_{1,1}-\sigma I - B_{1,2}(B_{2,2}-\sigma I)^{-1} B_{2,1})=0$, i.e., $\sigma$ is an eigenvalue of $B_{1,1}-B_{1,2}(B_{2,2}-\sigma I)^{-1} B_{2,1}$.

Using a tedious graph-theoretic argument [6, Appendix], it can be shown that the following is true.

**Lemma 2.** If $A_1$ is irreducible, then $B_{1,1}-B_{1,2}(B_{2,2}-tI)^{-1} B_{2,1}$ is irreducible for all $t \leqq \tilde{r}$. Moreover, for all $t \leqq \tilde{r}$, $B_{1,2}(B_{2,2}-tI)^{-1} B_{2,1}$ has at least one nonzero entry.

Let $g(\lambda) \equiv \varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-\lambda I)^{-1} B_{2,1}\right)$, with $0 \leqq \lambda \leqq \tilde{r}$. Since

$$B_{1,1}-B_{1,2}(B_{2,2}-\lambda I)^{-1} B_{2,1}$$

is a nonnegative irreducible matrix for all $\lambda$, $0 \leqq \lambda \leqq \tilde{r}$, it follows, from the Perron-Frobenius Theorem, that $g(\lambda)$ is a monotone increasing function of $\lambda$, $0 \leqq \lambda \leqq \tilde{r}$. Also, from the Perron-Frobenius Theorem, we know that $g(0) > 0$. Therefore, if $g(\tilde{r}) < \tilde{r}$ then, since $g$ is a continuous function of $\lambda$, it follows that there exists a $\tilde{\lambda}$ with $\tilde{\lambda} < \tilde{r}$ such that $\tilde{\lambda} = \varrho\left(B_{1,1} - B_{1,2}(B_{2,2} - \tilde{\lambda} I)^{-1} B_{2,1}\right)$. We now show that we do indeed have $g(\tilde{r}) < \tilde{r}$.

**Lemma 3.** If $g(\lambda) = \varrho\left(B_{1,1} - B_{1,2}(B_{2,2} - \lambda I)^{-1} B_{2,1}\right)$, then $g(\tilde{r}) < \tilde{r}$.

*Proof.* Let $E$ be a matrix of the same dimensions as $B_{2,1}$ but consisting entirely of ones. Consider $(B_{2,2} - \tilde{r} I) E + B_{2,1}$. Each entry of this matrix is greater than or equal to zero, since this is nothing more than the statement that

$$|a_{j,j}| - \max_{i \in S} \{|a_{i,i}| + \Lambda_i\} - \Lambda_j \geqq 0 \quad \text{for all} \quad j \notin S,$$

which is true by virtue of the fact that

$$|a_{j,j}| - |a_{i,i}| - \Lambda_i - \Lambda_j \geqq 0 \quad \text{for all} \quad i \in S$$

and all $j \notin S$. Since $B_{2,2} - \tilde{r} I$ is an $M$-matrix, we have that

$$E \geqq -(B_{2,2} - \tilde{r} I)^{-1} B_{2,1}.$$

Also, since $B_{1,2}$ is nonnegative we have that

$$B_{1,2} E \geqq -B_{1,2}(B_{2,2} - \tilde{r} I)^{-1} B_{2,1}.$$

Finally, we have that

$$B_{1,1} + B_{1,2} E \geqq B_{1,1} - B_{1,2}(B_{2,2} - \tilde{r} I)^{-1} B_{2,1} \equiv C(\tilde{r}). \tag{12}$$

Considering row sums on both sides of (12), we have that for each row $i$, $1 \leqq i \leqq s$,

$$|a_{i,i}| + \Lambda_i \geqq \mu_i, \tag{13}$$

where $\mu_i$ is the sum of the elements in the $i^{\text{th}}$ row of $C(\tilde{r})$. Since $\tilde{r} = \max_{i \in S} \{|a_{i,i}| + \Lambda_i\}$, we have that

$$\tilde{r} \geqq \mu_i \quad \text{for all} \quad i \in S. \tag{14}$$

Therefore, we have that

$$\tilde{r} \geqq \max_{i \in S} \mu_i \geqq \varrho\left(C(\tilde{r})\right). \tag{15}$$

However, from Lemma 2 and (11), equality in (15) would imply that $\tilde{r}$ is an eigenvalue of $C(\tilde{r})$ and hence an eigenvalue of $B$, which contradicts our assumptions on $B$. Therefore, $\tilde{r} > \varrho\left(C(\tilde{r})\right)$. But $\varrho\left(C(\tilde{r})\right) = g(\tilde{r})$, and the proof of the lemma is complete.

Let $\lambda$ be the greatest real number less than $\tilde{r}$ such that

$$\lambda = \varrho\left(B_{1,1} - B_{1,2}(B_{2,2} - \lambda I)^{-1} B_{2,1}\right).$$

We now wish to show that if $\tau$ is any eigenvalue of $A_{1,1} - A_{1,2}(A_{2,2} - \tau I)^{-1} A_{2,1}$, then $|\tau| \leqq \lambda$, i.e., if we can find $\lambda$, then $\lambda$ bounds the eigenvalues of $A_1$ which are in the isolated region.

**Lemma 4.** If $\lambda$ is the greatest real number less than $\tilde{r}$ such that

$$\lambda = \varrho\left(B_{1,1} - B_{1,2}(B_{2,2} - \lambda I)^{-1} B_{2,1}\right),$$

then $\lambda$ is a bound for all eigenvalues of $A_1$ in $|z| \leqq \tilde{r}$, i.e., all eigenvalues of $A_1$ in $|z| \leqq \tilde{r}$ are actually in $|z| \leqq \lambda$.

*Proof.* Let $\tau$ be an eigenvalue of $A_1$ in $|z| \leqq \tilde{r}$. Then

$$\det\left(A_{1,1} - A_{1,2}(A_{2,2} - \tau I)^{-1} A_{2,1} - \tau I\right) = 0$$

and therefore

$$|\tau| \leqq \varrho\left(A_{1,1} - A_{1,2}(A_{2,2} - \tau I)^{-1} A_{2,1}\right).$$

We then have the following inequalities:

$$\begin{aligned} |\tau| &\leqq \varrho(|A_{1,1} - A_{1,2}(A_{2,2} - \tau I)^{-1} A_{2,1}|) \\ &\leqq \varrho(|A_{1,1}| + |A_{1,2}|\,|(A_{2,2} - \tau I)^{-1}|\,|A_{2,1}|). \end{aligned}$$

But, by the definition of $B$,

$$\varrho(|A_{1,1}| + |A_{1,2}|\,|(A_{2,2} - \tau I)^{-1}|\,|A_{2,1}|) = \varrho\left(B_{1,1} - B_{1,2}|(A_{2,2} - \tau I)^{-1}|\, B_{2,1}\right).$$

Also, as a result of the definition of $B_{2,2}$ and Lemma 1, $A_{2,2} - \tau I$ is an $H$-matrix and therefore we have [7] that $|(A_{2,2} - \tau I)^{-1}| \leqq (B_{2,2} - |\tau| I)^{-1}$. It then follows that

$$|\tau| \leqq \varrho\left(B_{1,1} - B_{1,2}(B_{2,2} - |\tau| I)^{-1} B_{2,1}\right). \tag{16}$$

Since $\lambda$ is the greatest real number less than $\tilde{r}$ such that

$$\lambda = \varrho\left(B_{1,1} - B_{1,2}(B_{2,2} - \lambda I)^{-1} B_{2,1}\right),$$

it follows that $\lambda \geqq |\tau|$. Q.E.D.

**Lemma 5.** For $t_1 < t_2 \leqq \tilde{r}$, we have

$$-B_{1,2}(B_{2,2} - t_1 I)^{-1} B_{2,1} \leqq -B_{1,2}(B_{2,2} - t_2 I)^{-1} B_{2,1} \tag{17}$$

with strict inequality for at least one element.

*Proof.* Inequality (17) follows immediately from properties of $M$-matrices. The strict inequality for some entry follows from the second part of Lemma 2. Q.E.D.

## 3. Convergence Theorem

We now state and prove our main result which, in the special case $s = 1$, reduces to the result of Theorem 1 of [11].

**Theorem 1.** Let $A$ be an irreducible $n \times n$ matrix such that its first $s$, $1 \leqq s < n$, Gerschgorin disks are isolated. Assume $I(s) \neq \emptyset$ and let $K(\gamma, \tilde{r})$ be the unique minimum circle containing $\bigcup_{i=1}^{s} G_i(A)$. Let

$$A_1 = A - \gamma I = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \qquad \text{where } A_{1,1} \text{ is } s \times s.$$

Let $B$ be the comparison matrix for $A_1$, assuming also that not all the Gerschgorin circles associated with $B$ pass through $\tilde{r}$. Then, there exists a monotone decreasing sequence $\{\lambda_n\}_{n=0}^{\infty}$ converging to $\lambda$, where $\lambda$ is the eigenvalue of $B$ of greatest modulus in the region $|z| \leqq \tilde{r}$. Furthermore, $\lambda$ bounds the eigenvalues of $A_1$ in this region and therefore bounds the eigenvalues of $A$ in the region $|z - \gamma| \leqq \tilde{r}$.

*Proof.* Let

$$\lambda_{n+1} = \varrho\left(B_{1,1} - B_{1,2}(B_{2,2} - \lambda_n I)^{-1} B_{2,1}\right), \qquad n \geqq 0, \qquad \lambda_0 = \tilde{r}. \tag{18}$$

From the nonnegative irreducible character of the matrix

$$B_{1,1}-B_{1,2}(B_{2,2}-tI)^{-1}B_{2,1}$$

for $t\leqq\tilde{r}$ and Lemma 5, we know that for $t_2\leqq\tilde{r}$,

$$\varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-t_1I)^{-1}B_{2,1}\right)<\varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-t_2I)^{-1}B_{2,1}\right) \quad \text{if and only if} \quad t_1<t_2. \tag{19}$$

This inequality provides us with a basis for an induction on $\lambda_n$, i.e., as a result of inequality (19), if

$$\lambda_1=\varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-\lambda_0I)^{-1}B_{2,1}\right)<\lambda_0,$$

then $\lambda_{n+1}<\lambda_n$ for $n=0, 1, 2, \ldots$, and we will have a monotone strictly decreasing sequence $\{\lambda_n\}_{n=0}^{\infty}$ bounded below by $\varrho(B_{1,1})$ and hence a convergent sequence, i.e.,

$$\lim_{n\to\infty}\lambda_n=\lambda \quad \text{and} \quad \lambda=\varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-\lambda I)^{-1}B_{2,1}\right).$$

We, therefore, show that

$$\lambda_1=\varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-\lambda_0I)^{-1}B_{2,1}\right)<\lambda_0,$$

where $\lambda_0=\tilde{r}$. But this is nothing more than a restatement of Lemma 3, since $\lambda_1=g(\tilde{r})$ and $\lambda_0=\tilde{r}$. Thus $\lambda_0>\lambda_1$ and inductively $\lambda_{n+1}<\lambda_n$ for all $n\geqq 0$. Then

$$\lim_{n\to\infty}\lambda_n=\lambda \quad \text{and} \quad \lambda=\varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-\lambda I)^{-1}B_{2,1}\right).$$

The fact that $\lambda$ is an actual eigenvalue of $B$ follows from the fact that

$$B_{1,1}-B_{1,2}(B_{2,2}-\lambda I)^{-1}B_{2,1}$$

is a nonnegative irreducible matrix and thus the Perron Frobenius Theorem states that $\lambda$ is an eigenvalue of $B_{1,1}-B_{1,2}(B_{2,2}-\lambda I)^{-1}B_{2,1}$, i.e.,

$$\det\left(B_{1,1}-B_{1,2}(B_{2,2}-\lambda I)^{-1}B_{2,1}-\lambda I\right)=0.$$

The factorization (11) then shows that this implies that $\det(B-\lambda I)=0$. That this eigenvalue bounds the eigenvalues of $A_1$ in $|z|\leqq\tilde{r}$ and $A$ in $|z-\gamma|\leqq\tilde{r}$ is a result of Lemma 4 and the construction of $A_1$. Q.E.D.

## 4. The Class $\mathring{\Omega}_{A_1}$

Since many matrices would have the same comparison matrix $B$ associated with them, it is natural to ask what class of matrices is involved. Therefore, we let

$$\mathring{\Omega}_{A_1}\equiv\{C=(c_{i,j})\,|\,|c_{i,j}|=|a_{i,j}|,\ 1\leqq i,j\leqq n\}, \tag{20}$$

where $A_1=A-\gamma I=(a_{i,j})$. From the definition of the comparison matrix $B$, it is clear that each matrix $C\in\mathring{\Omega}_{A_1}$ has precisely the same comparison matrix and, furthermore, $B\in\mathring{\Omega}_{A_1}$, i.e., $\mathring{\Omega}_{A_1}=\mathring{\Omega}_B=\mathring{\Omega}_{|B|}$. It is also clear that each $C\in\mathring{\Omega}_{A_1}$ has $s$ eigenvalues in the disk $|z|\leqq\lambda$, where $\lambda$ is that eigenvalue of $B$ determined by Theorem 1. The goal of this section is to describe the spectrum of the class $\mathring{\Omega}_{A_1}$, restricted to $|z|\leqq\lambda$.

Let

(21) $S(\overset{\circ}{\Omega}_{A_1}) = \{z \mid 0 \leqq |z| \leqq \lambda \text{ and } C - zI \text{ is singular for some } C \in \overset{\circ}{\Omega}_{A_1}\}.$

Let

(22) $\Omega_{|A_1|} = \{C = (c_{i,j}) \mid c_{i,i} = |a_{i,i}|,\ 1 \leqq i \leqq n \text{ and } |c_{i,j}| = |a_{i,j}|,\ 1 \leqq i, j \leqq n\}.$

Let $G(\Omega_{|A_1|})$ be the minimal Gerschgorin set [3, 12] for $|A_1|$, i.e.,

$$G(\Omega_{|A_1|}) \equiv \bigcap_{\boldsymbol{x}>0} \left( \bigcup_{i=1}^{n} G_i(\boldsymbol{x}) \right), \tag{23}$$

where

$$G_i(\boldsymbol{x}) = \left\{ z \,\middle|\, |z - |a_{i,i}|| \leqq \frac{1}{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}|\, x_j \equiv \Lambda_i(\boldsymbol{x}) \right\}, \quad 1 \leqq i \leqq n. \tag{24}$$

$G(\Omega_{|A_1|})$ also contains all the eigenvalues of each $C$ belonging to $\Omega_{|A_1|}$. Let

$$\tilde{G}(\Omega_{|A_1|}) = \operatorname{rot} G(\Omega_{|A_1|}) \cap \{z \mid 0 \leqq |z| \leqq \lambda\}, \tag{25}$$

where $\operatorname{rot} G(\Omega_{|A_1|})$ is the rotation of the minimal Gerschgorin set about the origin, i.e., $z \in \operatorname{rot} G(\Omega_{|A_1|})$ if and only if $z\, e^{i\theta} \in G(\Omega_{|A_1|})$ for some real $\theta$. We remark that the effect of $\tilde{G}(\Omega_{|A_1|})$ is to restrict consideration to the disk $|z| \leqq \lambda$, just as $S(\overset{\circ}{\Omega}_{A_1})$, by definition, is restricted to the same disk. Now, if $\sigma \in S(\overset{\circ}{\Omega}_{A_1})$, then, since $\sigma \in \operatorname{rot} G(\Omega_{|A_1|})$ [4] and $|\sigma| \leqq \lambda$, we have that $\sigma \in \tilde{G}(\Omega_{|A_1|})$, i.e., $S(\overset{\circ}{\Omega}_{A_1}) \subseteq \tilde{G}(\Omega_{|A_1|})$.

We assume that $s > 1$, since the spectrum $S(\overset{\circ}{\Omega}_{A_1})$ has been characterized for the case where $s = 1$ in [11]. A dimensionality argument, together with a continuity argument, shows that we have the following [6]:

**Theorem 2.** There exists a nonnegative real number $\tau$ with $\tau < \lambda$ such that $\{z \mid 0 \leqq \tau \leqq |z| \leqq \lambda\} \subseteq S(\overset{\circ}{\Omega}_{A_1})$.

Under stronger hypotheses on $A_1$ we can prove considerably stronger results. As an example, we have the following theorem:

**Theorem 3.** Let $A_1$, $\lambda$, $S(\overset{\circ}{\Omega}_{A_1})$ and $\tilde{G}(\Omega_{|A_1|})$ be as in the previous discussion. Let $|A_1|$ be essentially diagonally dominant [4]. Then $S(\overset{\circ}{\Omega}_{A_1}) = \tilde{G}(\Omega_{|A_1|})$. Furthermore, if the $s$ isolated disks of $|A_1|$ are such that they cannot be isolated from each other, then

$$S(\overset{\circ}{\Omega}_{A_1}) = \tilde{G}(\Omega_{|A_1|}) = \{z \mid 0 \leqq \tau \leqq |z| \leqq \lambda\}, \quad \text{where} \quad \tau < \lambda.$$

*Proof.* In [4], it is shown that if $|A_1|$ is essentially diagonally dominant, then the spectrum of $\overset{\circ}{\Omega}_{|A_1|}$ is equal to $\operatorname{rot} G(\Omega_{|A_1|})$. Since $\tilde{G}(\Omega_{|A_1|})$ is merely the restriction of $\operatorname{rot} G(\Omega_{|A_1|})$ to the disk $|z| \leqq \lambda$ and $S(\overset{\circ}{\Omega}_{|A_1|}) = S(\overset{\circ}{\Omega}_{A_1})$ is the restriction of the spectrum of $\overset{\circ}{\Omega}_{|A_1|}$ to the same disk, we have that

$$S(\overset{\circ}{\Omega}_{A_1}) = \tilde{G}(\Omega_{|A_1|}). \tag{26}$$

We say that $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T \in P_s$ if and only if the first $s$ Gerschgorin disks of $X^{-1}|A_1|X$ are isolated from the remaining disks, where $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$.

We then say that the $s$ isolated disks of $|A_1|$ cannot be isolated from each other if for all $\boldsymbol{x}\in P_s$ and each proper subset $L$ of $S$, we have that

$$\Big(\bigcup_{i\in L} G_i(\boldsymbol{x})\Big)\cap\Big(\bigcup_{\substack{i\notin L\\ i\in S}} G_i(\boldsymbol{x})\Big)\neq\emptyset.$$

If we now assume that the $s$ isolated disks are such that they cannot be isolated from each other, then, using the definition of (23), it is not difficult to see that the minimal Gerschgorin set of $|A_1|$ must contain the segment of the real axis from some $\tau\geqq 0$ to $\lambda$ with $\tau\leqq\lambda$. Therefore, $\{z\,|\,0\leqq\tau\leqq|z|\leqq\lambda\}=\tilde{G}(\Omega_{|A_1|})$. From Theorem 2, we have that $\{z\,|\,0\leqq\tau\leqq|z|\leqq\lambda\}\subseteq S(\mathring{\Omega}_{A_1})$ and that $\tau<\lambda$. Combining this with (26) completes the proof. Q.E.D.

## 5. Numerical Example

In order to illustrate the algorithm of §3, we consider the following simple example. Let

$$A=\left[\begin{array}{cc|cc} 4 & 0 & 0 & 1\\ 0 & 4 & 1 & \frac{1}{2}\\ \hline \frac{1}{2} & \frac{1}{2} & -1 & 0\\ 1 & 1 & 0 & -2 \end{array}\right]. \tag{27}$$

For this matrix, $A_1$ is given by

$$A_1=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & \frac{1}{2}\\ \hline \frac{1}{2} & \frac{1}{2} & -5 & 0\\ 1 & 1 & 0 & -6 \end{array}\right], \tag{28}$$

and thus the comparison matrix $B$ is

$$B=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & \frac{1}{2}\\ \hline -\frac{1}{2} & -\frac{1}{2} & 5 & 0\\ -1 & -1 & 0 & 6 \end{array}\right]. \tag{29}$$

Using $\lambda_{n+1}=\varrho\left(B_{1,1}-B_{1,2}(B_{2,2}-\lambda_n I)^{-1}B_{2,1}\right)$ with $\lambda_0=\tilde{r}=\frac{3}{2}$, for this particular example, the first five iterates are 1.5, 0.4762, 0.3821, 0.3753, 0.3748 with all additional iterates being the same as the fifth iterate, correct to four decimal places.

JOHNSTON [2] considered an iterative technique similar to that of §3, but the method involved norms. His method is also applicable to this example. With $A$ partitioned as in (27) and using $l_2$-norms, the result of JOHNSTON's method is that $A$ has two eigenvalues belonging to the set $\{\sigma\,|\,|\sigma-4|\leqq 0.420\}$. With $A$ partitioned in the form

$$A=\left[\begin{array}{cc|c|c} 4 & 0 & 0 & 1\\ 0 & 4 & 1 & \frac{1}{2}\\ \hline \frac{1}{2} & \frac{1}{2} & -1 & 0\\ \hline 1 & 1 & 0 & -2 \end{array}\right], \tag{30}$$

and using $l_2$-norms, JOHNSTON's result is that $A$ has two eigenvalues belonging to the set $\{\sigma \,|\, |\sigma - 4| \leqq 0.439\}$. The corresponding result for the technique of this paper, as indicated by the iterates given above, is that $A$ has two eigenvalues belonging to the set $\{\sigma \,|\, |\sigma - 4| \leqq 0.375\}$.

*Added in Proof:* The case for which the first $s$ Gerschgorin disks can be isolated, under the stronger hypothesis that $a_{1,1} = a_{2,2} = \cdots = a_{s,s}$, has been recently studied by ELSNER [13].

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Dr. HELEN I. MEDLEY
Department of Mathematics
Kent State University
Kent, Ohio 44240
USA

Prof. Dr. RICHARD S. VARGA
Department of Mathematics
Case Western Reserve University
10900 Euclid Avenue
Cleveland, Ohio 44106/USA


Universitätsdruckerei H. Stürtz AG Würzburg