Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems
IV. Periodic Boundary Conditions*

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§ 1. Introduction

We shall consider here the numerical approximation of the solution of the following real nonlinear boundary value problem

\[ \mathcal{L}[u(x)] = f(x, u(x)), \quad 0 < x < 1, \]

with periodic boundary conditions

\[ D^k u(0) = D^k u(1), \quad 0 \leq k \leq 2n - 1, \quad D = \frac{d}{dx}, \]

where the differential operator \( \mathcal{L} \) is defined by

\[ \mathcal{L}[u(x)] = \sum_{j=0}^{n} (-1)^{j+1} D^j \phi_j(x) D^j u(x), \quad n \geq 1. \]

The coefficient functions \( \phi_j(x) \) are assumed to be of class \( C^1[0, 1] \) with periodic boundary behavior:

\[ D^k \phi_j(0) = D^k \phi_j(1), \quad 0 \leq k \leq j - 1, \quad 0 \leq j \leq n. \]

The material presented here is an extension of the results of [5, 7, 18] to such periodic boundary conditions (1.2), and it is convenient to follow the notations and assumptions given there. Here, in analogy with [6], let \( S \) denote the linear space of all real-valued functions \( w(x) \) defined on \( [0, 1] \) such that \( w(x) \in C^{n-1}[0, 1] \) with \( D^{n-1} w(x) \) absolutely continuous and \( D^n w(x) \in L^2[0, 1] \), and such that \( w(x) \) is \( 2n - 1 \) times continuously differentiable in neighborhoods of \( x = 0 \) and \( x = 1 \) of \( [0, 1] \) with

\[ D^k w(0) = D^k w(1), \quad 0 \leq k \leq 2n - 1. \]

Further, we assume that there exist two real constants \( K > 0 \) and \( \beta \) such that

\[ \|w\|_{L^n[0, 1]} = \sup_{x \in [0, 1]} |w(x)| \leq K \left( \int_0^1 \sum_{j=0}^{n} \phi_j(x) \left( D^j w(x) \right)^2 + \beta \left( w(x) \right)^2 \, dx \right)^{\frac{1}{2}} \]

for all \( w \in S \).

Next, we introduce the finite quantity (see Lemma 4)

\[ A \equiv \inf_{\|w\|_{L^n[0, 1]} = \beta} \left( \int_0^1 \sum_{j=0}^{n} \phi_j(x) \left( D^j w(x) \right)^2 \, dx \right)^{\frac{1}{2}} \]
Although this fact is not used, it turns out that \( A \) is a lower bound for the eigenvalues of the associated eigenvalue problem \( L[u(x)] + \lambda u(x) = 0, 0 < x < 1 \), subject to the boundary conditions of (1.2).

We finally assume that the function \( f(x, u) \) is real, continuous in both variables, i.e., \( f(x, u) \in C^0([0, 1] \times \mathbb{R}) \), that there exists a constant \( \gamma \) such that
\[
\frac{f(x, u) - f(x, v)}{u - v} \geq \gamma > -A.
\]
for all \( x \in [0, 1] \), and all \( -\infty < u, v < +\infty \), with \( u \neq v \), and for each \( c > 0 \), there exists a number \( M(c) \) such that
\[
(1.8') \quad u \neq v, \quad |u| \leq c, \quad |v| \leq c \quad \text{implies} \quad \frac{f(x, u) - f(x, v)}{u - v} \leq M(c) < \infty
\]
for all \( x \in [0, 1] \).

One of our main goals is to study the effects of applying the classical Rayleigh-Ritz procedure (cf. [6] and [10]) to a variational formulation [9] of (1.1)—(1.2) by minimizing over finite dimensional subspaces of \( S \) which for periodic problems are most naturally taken as subspaces of trigonometric polynomials. In so doing, we extend the results of [6] to periodic problems and obtain new error estimates which improve upon known results in the literature for the Galerkin Method, which is equivalent to the Rayleigh-Ritz procedure for the class of problems under consideration (cf. [3—5, 16, 17]).

Another of our goals is to show that these techniques can, from a numerical point of view, be efficiently applied on modern high-speed digital computers. To illustrate these theoretical results, numerical results for particular examples of (1.1)—(1.2) will be discussed.

§ 2. Rayleigh-Ritz Method

In this section, we discuss the Rayleigh-Ritz method for the problem (1.1) to (1.2). The proofs of the theorems of this section are exact analogues of the proofs given in [6] for the application of the Rayleigh-Ritz method to nonlinear two-point boundary value problems with Dirichlet boundary conditions. We begin with (cf. [6, Lemma 4])

**Lemma 1.** With the assumption of (1.6), then
\[
A = \inf_{w \in S} \left\{ \int \left( \sum_{j=0}^{n} p_j(x) [D^j w(x)]^4 \right) dx \right\} \geq \frac{1}{K^2} - \beta > -\infty.
\]

We make the essential hypothesis that (1.1)—(1.2) has a classical solution \( \varphi(x) \). Then, we have (cf. [6, Theorem 1]).

**Theorem 1.** With the assumptions of (1.4), (1.6), and (1.8), let \( \varphi(x) \) be a classical solution of (1.1)—(1.2). Then \( \varphi(x) \) strictly minimizes the following functional
\[
F[w] = \int \left( \frac{1}{2} \sum_{j=0}^{n} p_j(x) [D^j w(x)]^2 + \int_{0}^{w(x)} f(x, \eta) d\eta \right) dx
\]
over the space \( S \), and \( \varphi(x) \) is thus the unique classical solution of (1.1)—(1.2).
Consider now any finite dimensional subspace \( S_M \) of \( S \) of dimension \( M \), and let \( \{ w_i(x) \}_{i=1}^{M} \) be \( M \) linearly independent functions from the subspace. The analogue of Theorem 1, concerning the minimization of the functional \( F[w] \) over the subspace \( S_M \), is given in (cf. [6, Theorem 2])

**Theorem 2.** With the assumptions of (1.4), (1.6), and (1.8), there exists a unique function \( \widehat{w}_M(x) \) in the finite dimensional subspace \( S_M \) of \( S \) which minimizes the functional \( F[w] \) over \( S_M \).

To find this unique element \( \widehat{w}_M(x) = \sum_{i=1}^{M} \hat{u}_i w_i(x) \) in \( S_M \) which minimizes \( F[w] \) over \( S_M \), we must solve the \( M \) nonlinear equations

\[
(2.3) \quad \int_{0}^{1} \left( \sum_{j=0}^{n} \tilde{p}_j(x) \left( \sum_{k=1}^{M} u_k D^j w_k(x) \right) D^j w_i(x) + f(x, \sum_{k=1}^{M} u_k w_k(x)) w_i(x) \right) dx = 0, \quad 1 \leq i \leq M
\]

for the \( M \) unknowns \( u_1, u_2, \ldots, u_M \), which arise from

\[
\frac{\partial F}{\partial u_i} \left[ \sum_{k=1}^{M} u_k w_k \right] = 0, \quad 1 \leq i \leq M.
\]

Letting \( \mathbf{u} = [u_1, u_2, \ldots, u_M]^T \), the equations of (2.3) can be written in matrix form as

\[
(2.4) \quad A \mathbf{u} + \mathbf{g}(\mathbf{u}) = \mathbf{0},
\]

where \( A = (a_{i,k}) \) is a real \( M \times M \) matrix, and \( \mathbf{g}(\mathbf{u}) = [g_1(\mathbf{u}), g_2(\mathbf{u}), \ldots, g_M(\mathbf{u})]^T \) is a column vector, both being determined by

\[
(2.5) \quad a_{i,k} = \int_{0}^{1} \left( \sum_{j=0}^{n} \tilde{p}_j(x) D^j w_i(x) D^j w_k(x) \right) dx, \quad 1 \leq i, k \leq M,
\]

and

\[
(2.6) \quad g_i(\mathbf{u}) = \int_{0}^{1} f(x, \sum_{i=1}^{M} u_i w_i(x)) w_i(x) dx, \quad 1 \leq k \leq M.
\]

This nonlinear matrix equation (2.4) can then be efficiently solved for example by Gauss-Seidel iteration, which is known in this case to be convergent [12, 13]. For more computational details, we refer the reader to [6] and [8].

If we have a sequence \( \{ S_{M_i} \}_{i=1}^{\infty} \) of not necessarily nested finite dimensional subspaces of \( S \), we consider the problem of when the associated sequence \( \{ \widehat{w}_{M_i}(x) \}_{i=1}^{\infty} \) converges to \( \psi(x) \), the unique solution of (1.1)—(1.2), in the uniform norm. If \( \alpha \) is a real constant, define

\[
(2.7) \quad \| w \|_{\alpha} = \left\{ \int_{0}^{1} \left( \sum_{j=0}^{n} \tilde{p}_j(x) (D^j w(x)) \right)^2 + \alpha (w(x))^2 \right\}^{1/2} dx
\]

for all \( w \in S \). Then, recalling the constant \( \gamma \) of (1.8), we have (cf. [6, Lemma 2])

**Lemma 2.** If \( \alpha > -A \), then \( \| w \|_{\alpha} \) and \( \| w \|_{\beta} \) are equivalent norms on \( S \). Moreover, the inequality of (1.6) is valid for all \( w \in S \) with \( \beta \) replaced by any \( \gamma' \) with \( \gamma' > -A \).
As a consequence of this lemma, we can write for the particular constant $\gamma$ of (1.8) that

\begin{equation}
\|w\|_{L^\infty} \leq K \|w\|_r \quad \text{for all} \quad w \in S,
\end{equation}

which is an analogue of the original assumption of (1.6). In what is to follow, we shall regard $K$ and $\gamma$ as fixed constants satisfying (1.8) and (2.8).

The following fundamental result gives us an error estimate (cf. [6, Theorem 3]).

**Theorem 3.** Let $\psi(x)$ be the (classical) solution of (1.1)–(1.2), subject to the assumptions of (1.4), (1.6), and (1.8), let $S_M$ be any finite dimensional subspace of $S$, and let $\tilde{\psi}_M(x)$ be the unique function which minimizes $F[w]$ over $S_M$. Then, there exists a constant $C$ which is independent of the choice of $S_M$, such that the following error bound is valid:

\begin{equation}
\|\tilde{\psi}_M - \psi\|_{L^\infty} \leq K \|\tilde{\psi}_M - \psi\|_r \leq C \inf_{w \in S_M} \|w - \psi\|_r.
\end{equation}

As an immediate consequence of Theorem 3, we have

**Theorem 4.** Let $\psi(x)$ be the (classical) solution of (1.1)–(1.2), subject to the assumptions of (1.4), (1.6), and (1.8), let \(\{S_{M_i}\}_{i=1}^\infty\) be any sequence of (not necessarily nested) finite dimensional subspaces of $S$, and let \(\{\tilde{\psi}_{M_i}(x)\}_{i=1}^\infty\) be the sequence of functions obtained by minimizing $F[w]$ respectively over the subspaces $S_{M_i}$. If \(\lim_{i \to \infty} \{\inf_{w \in S_{M_i}} \|w - \psi\|_r\} = 0\), then \(\{\tilde{\psi}_{M_i}(x)\}_{i=1}^\infty\) converges uniformly to $\psi(x)$.

In place of \(\lim_{i \to \infty} \{\inf_{w \in S_{M_i}} \|w - \psi\|_r\} = 0\), we could of course make the stronger hypothesis in Theorem 4 that \(\lim_{i \to \infty} \{\inf_{w \in S_{M_i}} \|w - \psi\|_r\} = 0\) for every $g \in S$. In practice, where the solution $\psi(x)$ is in general not known a priori, this latter hypothesis is more readily checked, and is in fact valid for the subspaces considered in §3 and 4.

**§3. Trigonometric and Algebraic Polynomial Subspaces**

As our first example of subspaces $S_{M_i}$ satisfying the sufficient conditions of Theorem 4, let $N$ be any nonnegative integer, and let $T^{(N)}(N)$ be the collection of all real trigonometric polynomials $w(x)$ of order $N$, i.e.,

\begin{equation}
w(x) = \frac{a_0}{2} + \sum_{j=1}^{N} (a_j \cos(2\pi j x) + b_j \sin(2\pi j x)),
\end{equation}

where the coefficients $a_j$ and $b_j$ are real. It is clear that $T^{(N)}$ is a subspace of $S$ of dimension $2N+1$.

We now discuss the error of best least squares approximation by elements of $T^{(N)}$. For any function $f(x) \in L^2[0, 2\pi]$, let

\begin{equation}
\omega_2(f; \delta) = \sup_{|h| \leq \delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x + h) - f(x)|^2 dx \right\}^{\frac{1}{2}}
\end{equation}

denote the $L^2$-modulus of continuity, where $f(x)$ is periodically extended outside the interval $[0, 2\pi]$. We remark that $\omega_2(f; \delta)$ is a nondecreasing function of $\delta$, and that $\lim_{\delta \to 0} \omega_2(f; \delta) = 0$ for any $f(x) \in L^2[0, 2\pi]$. The following Jackson-like result is an easy extension of a result found in ALEXIT [2].
Theorem 5. For $s$ a positive integer, assume that $u(x) \in C^{s-1}_r[0, 2\pi]$, i.e., $u(x) \in C^{s-1}[0, 2\pi]$ and $D^s u(0) = D^s u(2\pi)$ for all $0 \leq j \leq s - 1$, and assume that $D^{i-1} u(x)$ is absolutely continuous with $D^i u(x) \in L^2[0, 2\pi]$. Then, for each positive integer $N$, there exists a trigonometric polynomial $\tilde{N}(x) \in T^N$ such that

$$
\|D^j(u - \tilde{N})\|_{L^2[0,2\pi]} \leq \frac{K \omega_2(D^s u; 1/N)}{N^{j-s}}
$$

for all $0 \leq j \leq s$, where $K$ is a constant dependent only on $s$.

**Proof.** We prove the case for $s = 1$; the cases $s > 1$ follow similarly by integration by parts. If $\frac{a_n}{2N} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ denotes the Fourier series for $u(x)$, then by integration by parts gives

$$
a_n = \frac{-1}{\pi} \int_0^{2\pi} Du(x) \sin nx \, dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_0^{2\pi} Du(x) \cos nx \, dx, \quad n \geq 1,
$$

where we have used the hypothesis that $u(0) = u(2\pi)$. Denoting the Fourier series for $Du(x) \in L^1[0, 2\pi]$ by $\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$, the constant term being necessarily zero since $u(x)$ is periodic, we see that $\alpha_n = nb_n$ and $\beta_n = -na_n$, $n \geq 1$, and thus

$$
|\alpha_n|^2 + |\beta_n|^2 = n^2 \left( |a_n|^2 + |b_n|^2 \right), \quad n \geq 1,
$$

and the best least squares approximation by trigonometric polynomials of order $N$ for $Du(x)$ is just $g_N(x) = \sum_{n=1}^{N} (\alpha_n \cos nx + \beta_n \sin nx)$. By Parseval's relation, we have

$$
\int_0^{2\pi} (Du(x) - g_N(x))^2 \, dx = \sum_{n=N+1}^{\infty} (|\alpha_n|^2 + |\beta_n|^2).
$$

On the other hand, we have from Alexits [2, p. 270] that

$$
\sum_{n=N+1}^{\infty} (|\alpha_n|^2 + |\beta_n|^2) \leq \frac{16}{3} \omega_2 \left(Du; \frac{1}{N}\right).
$$

Hence, if we define $\tilde{N}(x)$ by $\frac{a_n}{2N} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$, we see that $D\tilde{N}(x) = g_N(x)$, and combining (3.5) and (3.6) then yields

$$
\|D(u - \tilde{N})\|_{L^2[0,2\pi]} \leq \sqrt{\frac{16\pi}{3}} \omega_2 \left(Du; \frac{1}{N}\right)
$$

the special case $j = s = 1$ of (3.3). If we now compare $u(x)$ and $\tilde{N}(x)$, it follows from (3.4) that

$$
\int_0^{2\pi} (u(x) - \tilde{N}(x))^2 \, dx = \sum_{n=N+1}^{\infty} \left( |\alpha_n|^2 + |\beta_n|^2 \right) = \sum_{n=N+1}^{\infty} \frac{1}{n^2} \left( |\alpha_n|^2 + |\beta_n|^2 \right)
\leq \frac{16}{N^2} \sum_{n=N+1}^{\infty} \left( |\alpha_n|^2 + |\beta_n|^2 \right) \leq \frac{16}{3N^2} \omega_2 \left(Du; \frac{1}{N}\right),
$$

which gives the remaining inequality of (3.3) for the case $j = 0$, and $s = 1$. Q.E.D.
Corollary. With the hypotheses of Theorem 5, for each positive integer \( N \) there exists a trigonometric polynomial \( \tilde{l}_N(x) \) of order \( N \) such that

\[
(3.3') \quad \|D^j(u - \tilde{l}_N)\|_{L^1[0,2\pi]} \leq \frac{K'}{N^{s-j}} \|D^s u\|_{L^1[0,2\pi]} \quad \text{for all} \quad 0 \leq j \leq s,
\]

where \( K' \) is a constant dependent only on \( s \).

Proof. By definition, \( \omega_2(f; \delta) \leq \frac{2}{\sqrt{2\pi}} \|f\|_{L^1[0,2\pi]} \) for all \( \delta \geq 0 \). Q.E.D.

We now apply the result of Theorem 5 and its corollary to the solution \( \varphi(x) \) of (1.1)—(1.2), where we necessarily have that \( \varphi(x) \in C^s[0,1] \), and \( D^n \varphi \in L^2[0,1] \). The following result follows from the basic inequalities of (2.9) of Theorem 3.

**Theorem 6.** Let \( \varphi(x) \), the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class \( C^s[0,1] \) with \( D^n \varphi \in L^2[0,1] \), where \( s \geq 2n \), and let \( \tilde{l}_N(x) \) be the unique function which minimizes the functional \( F[w] \) over \( T(N) \).

Then, there exist constants \( M \) and \( M' \) depending only on \( s \) and \( \gamma \) such that

\[
\| \tilde{l}_N - \varphi \|_{L^\infty} \leq K \| \tilde{l}_N - \varphi \|_{L^1} \leq \frac{M}{N^{s-n}} \omega_2(D^s \varphi; \frac{1}{N}) \leq \frac{M'}{N^{s-n}} \| D^s \varphi \|_{L^1[0,1]}.
\]

If \( u(x) \in C^\infty_p[0,2\pi] \) is analytic in some open set of the complex plane containing the interval \([0,2\pi]\), we can of course apply the result of Theorem 5 for any \( s \geq 0 \). But, an even stronger result, indicating exponential convergence, is possible. As in Theorem 7 of [6], the following result can be established from a classical result of Bernstein (cf. [11, p. 158]).

**Theorem 7.** If \( u(x) \in C^\infty_p[0,2\pi] \) is analytic in some open set of the complex plane containing the interval \([0,2\pi]\), then there exist a constant \( \mu \) with \( 0 \leq \mu < 1 \) and a sequence of trigonometric polynomials \( \{\tilde{l}_N(x)\}_{N=0}^\infty \) with \( \tilde{l}_N(x) \in T(N) \) such that for each nonnegative integer \( n \),

\[
(3.8) \quad \lim_{N \to \infty} \left( \| D^k (u - \tilde{l}_N) \|_{L^\infty[0,2\pi]} \right)^{1/N} \leq \mu \quad \text{for all} \quad 0 \leq k \leq n.
\]

If \( u(x) \) is moreover an entire function, i.e., \( u(x) \) can be extended to a function \( u(z) \) which is analytic for all complex \( z \), then for each nonnegative integer \( n \),

\[
(3.8') \quad \lim_{N \to \infty} \left( \| D^k (u - \tilde{l}_N) \|_{L^\infty[0,2\pi]} \right)^{1/N} = 0 \quad \text{for all} \quad 0 \leq k \leq n.
\]

Applying this result to the solution \( \varphi(x) \) of (1.1)—(1.2), we have

**Theorem 8.** Let \( \varphi(x) \), the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class \( C^\infty_p[0,1] \) and be analytic in some open set in the complex plane containing the interval \([0,1]\), and let \( \tilde{l}_N(x) \) be the unique function which minimizes the functional \( F[w] \) over \( T(N) \). Then, there exists a constant \( \mu \) with \( 0 \leq \mu < 1 \) such that

\[
(3.9) \quad \lim_{N \to \infty} (\| \tilde{l}_N - \varphi \|_{L^\infty})^{1/N} = \mu,
\]

and consequently from Theorem 3,

\[
(3.9') \quad \lim_{N \to \infty} (\| \tilde{l}_N - \varphi \|_{L^\infty})^{1/N} \leq \mu.
\]

If \( \varphi(x) \) is moreover an entire function, the constant \( \mu \) can be chosen to be zero.
With respect to the solution of the nonlinear matrix problem (2.4) associated with the subspace $T^N$ of $S$, we mention that in special, but nevertheless interesting cases, it is easy to deduce explicitly an orthonormal basis for $T^N$, i.e., one for which the matrix entries $a_{i,j}$ of the matrix $A$ of (2.5) satisfy $a_{i,j} = \delta_{i,j}$. For example, for $n = 1$, $\tilde{p}_1(x) = 1$, $\tilde{p}_0(x) = 1$ in (1.3), the functions

$$
\left\{ 1; \left\{ \frac{2}{1 + 4\pi^2 n^2} \cos(2\pi n x), \frac{2}{1 + 4\pi^2 n^2} \sin(2\pi n x) \right\}_{n=1}^{N} \right\}
$$

form such a basis. This choice of basis makes the numerical solution of the nonlinear problem (2.4) considerably simpler.

As our second example of subspaces $S^N_M$ satisfying the sufficient conditions of Theorem 4, let $N$ be a positive integer with $N \geq 4n - 1$, and let $P^N_p$ be the collection of all real algebraic polynomials of degree $N$ which satisfy the periodicity condition of (1.5). It is clear that the elements of $P^N_p$ are polynomials which can be represented in the form

$$
b_0 h_0(x) + b_1 h_1(x) + \cdots + b_{2n-1} h_{2n-1}(x) + x^2 N(1 - x)^{2n} \left( a_0 + a_1 x + \cdots + a_{N-4n} x^{N-4n} \right),
$$

where $h_i(x)$, $0 \leq i \leq 2n - 1$, is the unique (Hermite) polynomial of degree $4n - 1$ such that

$$
D^j h_i(0) = D^j h_i(1) = \delta_{i,j}, \quad 0 \leq j \leq 2n - 4.
$$

It is clear that $P^N_p$ is a subspace of $S$ of dimension $N - 2n + 4$.

We now discuss the error in best approximation by elements of $P^N_p$.

**Theorem 9.** If $u(x) \in C^t[0, 1]$, $t \geq 2n$, and $u(x)$ satisfies the boundary conditions of (1.2), then for each positive integer $N \geq \max(t, 4n - 1)$, there exists an algebraic polynomial $\tilde{p}_N(x) \in P^N_p$ such that

$$
\|D^j(u - \tilde{p}_N)\|_{L^\infty} \leq K \frac{N}{(N - 2n)^{t+2n}} \omega \left( D^j u \left| \sum_{i=0}^{2n-1} D^j u(0) h_i(x) \right| \right) \frac{1}{N - 2n}
$$

for all $0 \leq j \leq 2n$, where $K$ is a constant dependent only on $t$ and $n$, and $\omega$ is the (usual) modulus of continuity.

**Proof.** Fixing $N \geq \max(t, 4n - 1)$, the proof is accomplished by applying Theorem 5 of [6] to the function $u(x) - \sum_{i=0}^{2n-1} D^j u(0) h_i(x)$, whose first $2n - 1$ derivatives are zero at $x = 0$ and $x = 1$. Q.E.D.

**Corollary.** With the hypotheses of Theorem 9, then for each positive integer $N \geq \max(t, 4n - 1)$, there exists an algebraic polynomial $\tilde{p}_N(x) \in P^N_p$ such that

$$
\|D^j(u - \tilde{p}_N)\|_{L^\infty} \leq \frac{K'}{(N - 2n)^{t+2n}} \left| \sum_{i=0}^{2n-1} D^j u(0) h_i(x) \right| \frac{1}{N - 2n}
$$

for all $0 \leq j \leq 2n$, where $K'$ is a constant dependent only on $t$ and $u$.

**Proof.** By definition, $\omega(f, \delta) = \sup_{|h| \leq \delta} |f(x + h) - f(x)| \leq 2\|f\|_{L^\infty}$. Q.E.D.
We remark that the result of Theorem 9 provides no estimates such as those of (3.3) for polynomials \( \tilde{p}_N(x) \) of low degree, i.e., for \( N < \max(t, 4n-1) \).

We now apply the result of Theorem 9 and its corollary to the solution \( \varphi(x) \) of (1.4)–(1.2), where we necessarily have that \( \varphi(x) \in C^4[0, 1] \). The following result follows from the basic inequalities of (2.9) of Theorem 3.

**Theorem 10.** Let \( \varphi(x) \), the solution of (1.4)–(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class \( C^1[0, 1] \) where \( t \geq 2n \), and let \( \tilde{p}_N(x) \) be the unique function which minimizes the functional \( F[w] \) over \( P_p^N \) where \( N \geq \max(t, 4n-1) \). Then, there exists constants \( M \) and \( M' \) depending only on \( t \) and \( \gamma \) such that

\[
\| \tilde{p}_N - \varphi \|_{L_\infty} \leq K \| \tilde{p}_N - \varphi \|_0,
\]

\[
\leq \frac{M}{(N-2n)^{t-2n}} \omega \left( D^t \varphi - \sum_{i=0}^{2n-1} (D^i \varphi(0)) \, h_i; \frac{1}{N-2n} \right),
\]

\[
\leq \frac{M'}{(N-2n)^{t-2n}} \left\| D^t \varphi - \sum_{i=0}^{2n-1} (D^i \varphi(0)) \, h_i \right\|_{L_\infty}.
\]

If \( u(x) \) is actually analytic in some open set of the complex plane containing the interval \([0, 1]\), we can apply the result of Theorem 9 for any \( t \geq 2n \). However, an even stronger result, indicating exponential convergence, is possible. As in Theorem 7 of [6], which is based on a classical result of Bernstein (cf. [11, p. 162]), the following can be established.

**Theorem 11.** Let \( u(x) \) be analytic in some open set of the complex plane containing the interval \([0, 1]\). Then, there exists a constant \( \mu \) with \( 0 \leq \mu < 1 \), and a sequence of algebraic polynomials \( \{ \tilde{p}_N(x) \}_{N=4n-1}^\infty \) with \( \tilde{p}_N(x) \in P_p^N \) such that

\[
\lim_{N \to \infty} \left( \| D^k (u - \tilde{p}_N) \|_{L_\infty} \right)^{1/N} \leq \mu \quad \text{for all} \quad 0 \leq k \leq n.
\]

Applying this result to the solution \( \varphi(x) \) of (1.4)–(1.2), we then have

**Theorem 12.** Let \( \varphi(x) \), the solution of (1.4)–(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be analytic in some open set of the complex plane containing the interval \([0, 1]\), and \( \tilde{p}_N(x) \) be the unique function which minimizes the functional \( F[w] \) over \( P_p^N \) where \( N \geq 4n - 1 \). Then, there exists a constant \( \mu \) with \( 0 \leq \mu < 1 \) such that

\[
\lim_{N \to \infty} \left( \| \tilde{p}_N - \varphi \|_{L_\infty} \right)^{1/N} = \mu,
\]

and consequently from Theorem 3,

\[
\lim_{N \to \infty} \left( \| \tilde{p}_N - \varphi \|_0 \right)^{1/N} \leq \mu.
\]

§ 4. \( L \)-Splines and \( G \)-Splines

To give other examples of subspaces \( S_M \), satisfying the sufficient conditions of Theorem 4, we consider now subspaces of \( L \)-splines, which include the Hermite and natural spline subspaces as special cases (cf. [15]). For each positive integer \( m \), let \( K^m_p[0, 1] \) denote the collection of all real-valued functions \( u(x) \) defined on
\[ [a, b] \text{ such that } u(x) \in C^{m-1}[a, b] \text{ and such that } D^{m-1}u(x) \text{ is absolutely continuous with } D^m u(x) \in L^2[0, 1]. \] Assuming \( m \geq n \), let \( L \) be the \( m \)-th order differential operator defined by
\[
L[u(x)] = \sum_{j=0}^{n} a_j(x) D^j u(x)
\]
for any \( u(x) \in C^n[a, \bar{b}] \), where \( a_j(x) \in K^n[a, b] \) for all \( 0 \leq j \leq m \), and \( a_m(x) \geq \omega > 0 \) for all \( x \in [0, 1] \). Let \( \pi: 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1 \) denote a partition of \( [0, 1] \), let \( z = (z_1, z_2, \ldots, z_N) \) denote an incidence vector associated with \( \pi \) where the components \( z_i \) are positive integers satisfying \( 1 \leq z_i \leq m \) for all \( 1 \leq i \leq N \), and let \( L^{\pi} \), the formal adjoint of \( L \), be defined by \( L^{\pi}[v(x)] = \sum_{j=0}^{m} (-1)^j D^j(a_j(x) v(x)) \).

Then, the collection of all real-valued functions \( s(x) \) such that \( s(x) \in K^m [x_i, x_{i+1}] \) for each \( 0 \leq i \leq N \), satisfying
\[
L^{\pi} L[s(x)] = 0 \quad \text{for almost all } x \in (x_i, x_{i+1}), \quad 0 \leq i \leq N,
\]
and
\[
D^k s(x_i-) = D^k s(x_i+) \quad \text{for } 0 \leq k \leq 2m - 1 - z_i, \quad 1 \leq i \leq N,
\]
with
\[
D^k s(0) = D^k s(1) \quad \text{for } 0 \leq k \leq 2n - 1,
\]
will be denoted by \( S^\pi_p(L, \pi, z) \). Because \( m \geq n \), then \( S^\pi_p(L, \pi, z) \) is readily seen to be a finite-dimensional subspace of \( \mathcal{S} \). As a special case, the choice \( L = L^{\pi} = D^m \) is such that each element \( s(x) \in S^\pi_p(L, \pi, z) \) is a piecewise-polynomial function in \( [0, 1] \), i.e., \( s(x) \) is a polynomial of degree \( 2m - 1 \) in each interval \([x_i, x_{i+1}]\), \( 0 \leq i \leq N \). If, in addition, the components \( z_i \) of the incidence vector \( z \) are all chosen to be \( m \), we obtain in particular the (smooth) periodic Hermite space \( H^m_p(\pi) \). Such subspaces were used in our practical computations, to be described in \( \S \) 5.

Given any real-valued function \( f(x) \in C^{m-1}[0, 1] \) such that \( D^k f(0) = D^k f(1) \) for \( 0 \leq k \leq 2n - 1 \), and any partition \( \pi \) of \( [0, 1] \), and any associated incidence vector \( z \), there exists a (unique) element (cf. \[15, \text{Theorem 3}\]) \( \bar{s}(x) \in S^\pi_p(L, \pi, z) \) such that
\[
D^k \bar{s}(x_i) = D^k f(x_i), \quad 0 \leq k \leq z_i - 1, \quad 1 \leq i \leq N,
\]
\[
D^k \bar{s}(x_i) = D^k f(x_i), \quad 0 \leq k \leq m - 1, \quad i = 0 \quad \text{and} \quad i = N + 1.
\]
This corresponds to interpolation of Type I in \[15\]. Similarly, if \( f(x) \in C^{2m-1}_p[0, 1] \) and \( a_j(x) \in C^{m-1}_p[0, 1] \) for all \( 0 \leq j \leq m \), then for any sufficiently fine partition \( \pi \) of \([0, 1]\), i.e., for \( \bar{n} = \max |x_{i+1} - x_i| \) sufficiently small, and any associated incidence vector \( z \), there exists a (unique) element \( \bar{s}(x) \in S^\pi_p(L, \pi, z) \) such that
\[
D^k \bar{s}(x_i) = D^k f(x_i), \quad 0 \leq k \leq z_i - 1, \quad 1 \leq i \leq N,
\]
\[
D^k \bar{s}(x_i) = D^k f(x_i), \quad 0 \leq k \leq z_i - 1, \quad i = 0, \quad \text{and}
\]
\[
D^k \bar{s}(0) = D^k \bar{s}(1), \quad z_0 \leq k \leq 2m - 1,
\]
for any positive integer \( z_0 \) with \( 1 \leq z_0 \leq m \). This corresponds to interpolation of Type IV in [15]. For brevity, we shall in the following consider only the interpolation of (4.5). Note that the classical solution \( q(x) \) of (1.1) \(-\) (1.2) for \( 2n \geq m - 1 \) then always possesses a unique interpolate \( \tilde{s}(x) \) in \( S_{ph}(L, \pi, z) \) satisfying (4.5) for any choice of partition \( \pi \) and associated incidence vector \( z \).

It follows from Theorem 9 of [15] that if \( q(x) \in K^2_{\pi}[0, 1] \), then, for any partition \( \pi \) of \([0, 1]\) and any associated incidence vector \( z \), there exists a constant \( M \), dependent only on \( j \) and \( m \), such that

\[
D^j (q - \tilde{s})\|_{L^p[0,1]} \leq M(\pi)^{2-n-j} \|L^p L q\|_{L^p[0,1]} \quad \text{for} \quad 0 \leq j \leq m,
\]

where \( q(x) \) is the classical interpolation of (1.1) \(-\) (1.2), and \( \tilde{s}(x) \) is its unique interpolation (in the sense of (4.5)) in \( S_{ph}(L, \pi, z) \). As a direct consequence of this inequality and the basic inequalities of (2.9) of Theorem 3, we have

**Theorem 13.** Let \( q(x) \), the solution of (1.1) \(-\) (1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class \( K^2_{\pi}[0, 1] \), and let \( \tilde{s}(x) \) be the unique function which minimizes the functional \( F[w] \) over \( S_{ph}(L, \pi, z) \). Then, there exists a constant \( M \) depending only on \( m \) and \( \gamma \) such that

\[
\|\tilde{s} - q\|_{L^p[0,1]} \leq K \|\tilde{s} - q\|_{L^0} \leq M(\pi)^{2-n} \|L^p L q\|_{L^p[0,1]}.
\]

As a direct consequence of the inequality of (4.8), we have the

**Corollary.** Let \( q(x) \), the solution of (1.1) \(-\) (1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class \( K^2_{\pi}[0, 1] \), let \( \{\pi_i\}_{i=1}^{\infty} \) be any sequence of partitions of \([0, 1]\) with \( \lim_{i \to \infty} \pi_i = 0 \), and let \( \{z_i\}_{i=1}^{\infty} \) be any sequence of associated incidence vectors. If \( \tilde{s}_i(x) \) is the unique function which minimizes the functional \( F[w] \) over \( S_{ph}(L, \pi_i, z_i) \), then \( \{\tilde{s}_i(x)\}_{i=1}^{\infty} \) converges uniformly to \( q(x) \).

In the special case that \( L[\pi](x) = \text{const} \) for \( x \in [0, 1] \), the previous results may be further generalized. As before, let \( \pi: 0 = x_0 < x_1 < \cdots < x_{N+1} = 1 \) denote a partition of \([0, 1]\), and let \( E = (\epsilon_{i,j}) \) denote an \( N \times m \) incidence matrix, \( 1 \leq i \leq N \), \( 0 \leq j \leq m - 1 \), having entries of 0's or 1's, with at least one nonzero entry in each row of \( E \). Further, let \( \epsilon \) denote the collection of \( (i,j) \) such that \( \epsilon_{i,j} = 1 \). Then, from the results of [1, 14], and [15], \( S_{ph}(m, \pi, E) \) denote the collection of all real-valued functions \( s(x) \), called periodic \( \gamma \)-splines of order \( m \) for \( \pi \) and \( E \), defined on \([0, 1]\) such that

\[
s(x) \text{ is a polynomial of degree at most } 2m - 1 \text{ in each subinterval } (x_i, x_{i+1}),
\]

\[
0 \leq i \leq N,
\]

and \( D^m s(0) = D^m s(1) \quad \text{for} \quad 0 \leq k \leq 2m - 1.\)

Given any real-valued function \( f(x) \in C^{m-1}[0, 1] \) such that \( D^m f(0) = D^m f(1) \) for \( 0 \leq \epsilon \leq 2m - 1 \), any partition \( \pi \) of \([0, 1]\) and any incidence matrix \( E \), there exists a (unique) element (cf. [15, Theorem 15]) \( s(x) \in S_{ph}(m, \pi, E) \) such that

\[
D^i \tilde{s}(x_i) = D^i f(x_i) \quad \text{for all } (i,j) \in \epsilon,
\]

\[
D^m \tilde{s}(x_i) = D^m f(x_i), \quad 0 \leq k \leq m - 1, \quad i = 0 \quad \text{and} \quad i = N + 1.
\]
This corresponds to interpolation of Type I in [15]. As before, other types of interpolation are possible, and we refer the reader to [15].

Let \( \{\pi_i\}_{i=1}^{\infty} \) and \( \{E^{(0)}\}_{i=1}^{\infty} \) be a sequence of partitions of \([0, 1]\) and an associated sequence of incidence matrices such that if \( \pi_i: 0 = x_0^{(i)} < x_1^{(i)} < \cdots < x_{N_i+1}^{(i)} = 1 \), then we assume as in [1] and [15] that there exists a positive constant \( c \) such that for each \( k \) with \( 0 \leq k \leq N_i + 1 \) there exists an integer \( j = j(k) \) such that \( e_j^{(i)} = 1 \) and

\[
(x_j^{(i)} - x_i^{(i)}) \leq c \bar{z}_i, \quad \text{for all } i \geq 1, \quad \text{all } 0 \leq k \leq N_i + 1.
\]

(4.11)

It then follows from Theorem 21 of [15] that if \( q(x) \in C^2[0, 1] \), then there exists a constant \( M \), dependent only on \( j \) and \( m \), such that

\[
\|D^j(q - \bar{z}_i)\|_{L^2} \leq M(\bar{z}_i)^{2m-j}\|D^{2m}q\|_{L^2} \quad \text{for any } 0 \leq j \leq m,
\]

(4.12)

where \( q(x) \) is the classical solution of (1.1)–(1.2), and \( \bar{z}_i(x) \) is its unique interpolation (in the sense of (4.10)) in \( S_{P_p}(m, \pi_i, E_j) \). As a direct consequence of this inequality and the basic inequalities of (2.9) of Theorem 3, we have

**Theorem 14.** Let \( q(x) \), the solution of (1.1)–(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class \( C^2[0, 1] \), and let \( \bar{z}_i(x) \) be the unique function which minimizes the functional \( F[w] \) over \( S_{P_p}(m, \pi_i, E_j) \), where the partitions \( \{\pi_i\}_{i=1}^{\infty} \) and associated incidence matrices \( \{E^{(0)}\}_{i=1}^{\infty} \) satisfy the hypothesis of (4.11). Then, there exists a constant \( M \), depending only on \( m \) and \( \gamma \), such that

\[
\|\bar{z}_i - q\|_{L^\infty} \leq K \|\bar{z}_i - q\|_p \leq M(\bar{z}_i)^{2m-n}\|D^{2m}q\|_{L^2} \quad \text{for all } i \geq 1.
\]

(4.13)

If, in addition, \( \lim_{i \to \infty} \bar{z}_i = 0 \), then the sequence \( \{\bar{z}_i(x)\}_{i=1}^{\infty} \) converges uniformly to \( q(x) \).

**5. Numerical Results**

In this section, we discuss the numerical results we have obtained for some concrete examples by using particular subspaces described in §§ 3–4 in the Rayleigh-Ritz procedure. Let us however first summarize the results of §§ 3–4 by comparing the asymptotic error estimates for the subspaces \( T^{(N)} \) and \( S_{P_p}(L, \pi, \alpha) \) in terms of the total number of parameters associated with each of the subspaces.

Let \( q(x) \), the solution of (1.1)–(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class \( C^2[0, 1] \) with \( D^s q \in L^2(0, 1) \), where \( s \geq 2m \). If \( \bar{z}_N(x) \) is the unique function which minimizes \( F[w] \) over the subspace \( T^{(N)} \), then (cf. Theorem: 6)

\[
\|\bar{z}_N - q\|_{L^\infty} = \epsilon \left( \frac{1}{(d_N - 1)\gamma} \right)^{-n} \quad \text{as } N \to \infty, \quad d_N = 2N + 1,
\]

(5.1)

where \( d_N \) is the dimension of \( T^{(N)} \). Next, consider a sequence of (smooth) periodic Hermite subspaces \( \{H^{(m)}_{P_p}(\pi_i)\}_{i=1}^{\infty} \) where \( m \) is fixed. As previously noted, \( H^{(m)}_{P_p}(\pi_i) \) corresponds to the periodic \( L \)-spline subspace \( S_{P_p}(D^m, \pi_i, \pi^{(0)}) \) where each component of the associated incidence vector \( \pi^{(0)} \), \( x_j^{(i)} \), satisfies \( x_j^{(i)} = m \). Such subspaces were used in the actual numerical computations to be described. We assume that the partitions \( \pi_i: 0 = x_0^{(i)} < x_1^{(i)} < \cdots < x_{N_i+1}^{(i)} = 1 \) satisfy the regularity conditions that

\[
\bar{z}_i \leq K \quad \text{for all } i \geq 1, \quad \text{and } \lim_{i \to \infty} N_i = + \infty.
\]

(5.2)
If \( \hat{u}_i(x) \) is the unique function which minimizes the functional \( F[w] \) over \( H^m_f(\pi_i) \), then (cf. Theorem 13)

\[
\left\| \hat{u}_i - \varphi \right\|_{L^\infty} = \Theta \left\{ \frac{1}{(d_i - m + 2n)^{2m - n}} \right\}, \quad d_i = m(N_i + 2) - 2n, \quad i \to \infty,
\]

where \( d_i \) is the dimension of \( H^m_f(\pi_i) \). Thus, in the special case that \( s = 2m \), we observe, surprisingly enough, that these theoretical error estimates are asymptotically the same.

We now consider the numerical solution of particular examples of (1.4)–(1.2). As our first example, consider the linear differential equation

\[
D^4 u(x) = u(x) + \sin(2\pi x) \left\{ \left( -1 - 4\pi^2 \right) \left( x^2 - \frac{4}{3}x^2 + \frac{x}{3} \right) + 6x - \frac{8}{3} \right\} \cos(2\pi x) \left\{ 3x^2 - \frac{8}{3}x + \frac{1}{3} \right\}, \quad 0 < x < 1,
\]

subject to the boundary conditions of

\[
u(0) = u(1), \quad D u(0) = D u(1).
\]

For this example, we have \( f_1(x) = 1, \ f_2(x) = 0 \) in (1.3), and thus (1.4) is satisfied. By virtue of Sobolev’s inequality in one dimension [19, p. 174], we have that there exists a positive constant \( K \) such that

\[
\left\| w \right\|_{L^\infty} \leq \left\{ \int_0^1 \left[ \left( D w(t) \right)^3 + \left( w(t) \right)^3 \right] dt \right\}^{\frac{1}{3}} \leq K \left\| w \right\|_{L^\infty} \text{ for all } w(x) \in S.
\]

Thus, we see that the inequality of (1.6) is satisfied for the choice \( \beta = +1 \). Next, we see that the quantity \( \lambda \) of (1.7) is necessarily zero for this example, and as \( f(x, u) = u + g(x) \), then \( f_u = +1 \), showing that we can choose \( \gamma = +1 \). Thus, the inequality of (1.8) is satisfied. The unique solution \( \psi(x) \) of (5.4)–(5.5) is given by

\[
\varphi(x) = \left( x^3 - \frac{4}{3}x^2 + \frac{x}{3} \right) \sin(2\pi x), \quad 0 \leq x \leq 1.
\]

In this case, \( \varphi(x) \in C^1[0, 1] \). Since \( C^1[0, 1] \), and Theorem 6 is applicable with \( s = 2 \) and \( n = 1 \).

The numerical results of minimizing the associated functional \( F[w] \) over the trigonometric subspaces \( T^{(n)} \) are summarized in Table 5.1. For specific details about the efficient numerical minimization of \( F[w] \) over various subspaces, we refer the reader to [6] and [8].

For purposes of comparison, numerical results were also obtained for minimizing the functional \( F[w] \) over the (smooth) periodic cubic Hermite subspaces

| \begin{array}{|c|c|c|} \hline N \text{ in } T^{(N)} & \text{dim } T^{(N)} & \left\| \hat{u}_N - \varphi \right\|_{L^\infty} \\
\hline 4 & 9 & 9.63 \cdot 10^{-4} \\
6 & 13 & 3.98 \cdot 10^{-4} \\
8 & 17 & 2.17 \cdot 10^{-4} \\
10 & 21 & 1.35 \cdot 10^{-4} \\
12 & 25 & 9.35 \cdot 10^{-5} \\
\hline \end{array} |

| \begin{array}{|c|c|c|} \hline k & \text{dim } H^{(2)}(\pi(k)) & \left\| \hat{u}_k - \varphi \right\|_{L^\infty} \\
\hline \hat{k} & 10 & 7.48 \cdot 10^{-4} \\
\hat{k} & 14 & 2.21 \cdot 10^{-4} \\
\hat{k} & 20 & 6.42 \cdot 10^{-5} \\
\hat{k} & 32 & 1.10 \cdot 10^{-5} \\
\hat{k} & 50 & 1.97 \cdot 10^{-6} \\
\hline \end{array} |

Table 5.1. Trigonometric subspaces \( T^{(N)} \) | Table 5.2. Smooth periodic Hermite subspaces \( H^{(2)}_f(\pi(k)) \)
\[ H^N_p(\pi(h)) \], where a uniform partition \( \pi(h) \) of \([0, 1] \) with mesh-size \( h \) was used. The results are summarized in Table 5.2. For this case, Theorem 13 is applicable with \( m = 2 \) and \( n = 1 \).

As our second example, consider the \textit{nonlinear} differential equation

\begin{equation}
D^2 u(x) = u(x) + (u(x))^3 + e^{\sin(2\pi x)} [4\pi^2 \cos^2(2\pi x) - 4\pi^2 \sin(2\pi x) - 1], \quad 0 < x < 1,
\end{equation}

subject to the boundary conditions of

\begin{equation}
u(0) = u(1), \quad D u(0) = D u(1).
\end{equation}

As in the previous example, we have \( p_1(x) = 1 \), \( p_0(x) = 0 \) in (1.3), and thus (1.4) is again satisfied. From the inequality of (5.6), we again have that the inequality of (1.6) is satisfied for the choice \( \beta = 1 \), and A of (1.7) is again zero. For this example, \( f(x, u) = u + u^3 + g(x) \), so that \( f_u(x, u) = 1 + 3u^2 \geq 1 \), and again the inequality of (1.8) is satisfied for the choice \( \gamma = 1 \). The unique solution of (5.8)\(\rightarrow\)(5.9) is given by

\begin{equation}
\varphi(x) = e^{\sin(2\pi x)}, \quad 0 \leq x \leq 1.
\end{equation}

Thus, for this case \( \varphi(x) \in C^\infty_p[0, 1] \) and \( \varphi(x) \) can be extended to an entire function (cf. § 3). Thus, Theorem 8 is applicable with \( \mu = 0 \), and we have

\begin{equation}
\lim_{N \to \infty} \left( \| \hat{P}_N - \varphi \|_\infty \right)^{1/N} = 0.
\end{equation}

The numerical results of minimizing the associated functional \( F[w] \) over the trigonometric subspaces \( T^{(N)} \) are summarized in Table 5.3.

<table>
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<th>( N ) in ( T^{(N)} )</th>
<th>( \dim T^{(N)} )</th>
<th>( | \hat{a}<em>N - \varphi |</em>\infty )</th>
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</tr>
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</table>

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References


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