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Discrete Variational Green's Function

II. One Dimensional Problem

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§ 1. Introduction

Let L denote as in [6], the $2n$-th order real differential operator defined by

$$(1.1) \qquad L\,u(x) = \sum_{j=0}^{n} (-1)^j D^j\{p_j(x)\, D^j u(x)\}, \qquad 0 < x < 1,$$

where n is some positive integer, and $D^k \equiv d^k/dx^k$. Throughout this paper, we shall assume that

$$(1.2) \qquad p_j \in C^{2j}[0, 1] \qquad \text{for all} \quad 0 \leqq j \leqq n,$$

and that

$$(1.3) \qquad p_n(x) \geqq \omega > 0 \qquad \text{for all} \quad 0 \leqq x \leqq 1,$$

where ω is a constant. With $(.,.)$ denoting the usual inner product in $L^2[0, 1]$ and $\|.\|$ denoting the corresponding norm, i.e.,

$$(u, v) = \int_0^1 u(\xi)\, v(\xi)\, d\xi, \qquad \text{and} \qquad \|u\| = (u, u)^{\frac{1}{2}},$$

the Sobolev space $W_0^{n,2}[0, 1]$ then denotes the collection of all real-valued functions u defined on $[0, 1]$ such that $u \in C^{n-1}[0, 1]$ with $D^{n-1}u$ absolutely continuous and $D^n u \in L^2[0, 1]$, and which satisfy the Dirichlet boundary conditions

$$(1.4) \qquad D^k u(0) = D^k u(1) = 0 \quad \text{for all} \quad 0 \leqq k \leqq n-1.$$

Writing $S \equiv W_0^{n,2}[0, 1]$ for brevity, the space S is a Hilbert space, with inner product defined by

$$(1.5) \qquad (u, v)_S = \sum_{j=0}^{n} (D^j u, D^j v), \qquad u, v \in S,$$

and norm defined by

$$(1.5') \qquad \|u\|_S^2 = \sum_{j=0}^{n} \|D^j u\|^2.$$

In what follows, we make use of the fact that positive constants α_k, β_l exist such that for all $u \in S$

$$(1.6) \qquad \|D^k u\| \leqq \alpha_k \|u\|_S \quad \text{for all} \quad 0 \leqq k \leqq n,$$

and

$$(1.7)\qquad \|D^l u\|_\infty \equiv \sup\{|D^l u(x)|;\ 0 \leqq x \leqq 1\} \leqq \beta_l \|u\|_S \quad \text{for all} \quad 0 \leqq l \leqq n-1.$$

We now associate with the differential operator L of (1.1) the symmetric bilinear form

$$(1.8)\qquad a(u, v) = \int_0^1 \left\{\sum_{j=0}^n p_j(x)\, D^j u(x)\, D^j v(x)\right\} dx$$

which is defined for all $u, v \in S$. It follows from the assumptions of (1.2) and the inequalities of (1.6) that the bilinear form $a(.,.)$ is *continuous* over $S \times S$, i.e., there exists a constant K_1 such that

$$(1.9)\qquad |a(u, v)| \leqq K_1 \|u\|_S \|v\|_S, \quad \text{for all} \quad u, v \in S.$$

Finally, we assume that the bilinear form $a(.,.)$ is (strongly) *coercive* in the sense that there exists a *positive* constant K_2 such that

$$(1.10)\qquad K_2(\|u\|_S)^2 \leqq a(u, u) \quad \text{for all} \quad u \in S.$$

Using the case $k=0$ of (1.6) and the Rayleigh-Ritz characterization of the eigenvalues of L, the assumption of (1.10) gives us that all eigenvalues λ of $Lu = \lambda u$, subject to the boundary conditions of (1.4), are positive. Combining (1.9) and (1.10),

$$K_2(\|u\|_S)^2 \leqq a(u, u) \leqq K_1(\|u\|_S)^2,$$

it follows that

$$(1.11)\qquad \{a(u, u)\}^{\frac{1}{2}}$$

is a *norm* on the space S, equivalent to the norm $\|u\|_S$ of (1.5′), and

$$(1.12)\qquad [u, v] \equiv a(u, v)$$

is an *inner-product* over the space S.

With the assumptions and notations just made, we now give an outline of the material to be presented in the subsequent sections. In § 2, we introduce the discrete variational Green's function $G^N(x, \xi)$ which is such that the variational approximation Φ^N of the solution Φ of

$$(1.13)\qquad Lu(x) = f(x), \quad 0 < x < 1, \quad \text{with} \quad D^k u(0) = D^k u(1) = 0, \quad 0 \leqq k \leqq n-1,$$

relative to the subspace S^N of S (cf. (2.17)) is given by

$$(1.14)\qquad \Phi^N(x) = \int_0^1 G^N(x, \xi) f(\xi)\, d\xi, \quad 0 \leqq x \leqq 1.$$

In the next sections, we then *specialize* the subspaces S^N of S, and compare the function $G^N(x, \xi)$ with the usual Green's function $G(x, \xi)$ for the operator L. In particular, in § 4 we compare there Green's functions for Hermite L-spline subspaces of S.

§ 2. The Discrete Variational Green's Function

Let us first recall a few relevant and classical properties of the Green's function associated with the operator L of (1.1) and the boundary conditions of (1.4).

Theorem 1. With the assumptions of §1, there exists a *Green's function* $G(x, \xi)$ defined on the closed unit square $0 \leqq x, \xi \leqq 1$, which possesses the following properties:

(2.1) $G(x, \xi) = G(\xi, x)$ in the closed unit square,

(2.2) $G(x, \xi) \in C^{2n-2}$ in the closed unit square,

(2.3) $G(x, \xi) \in C^{2n}$ in the subsets $0 \leqq x < \xi \leqq 1$ and $0 \leqq \xi < x \leqq 1$ of the closed unit square.

Given any $0 \leqq \xi \leqq 1$, let G_ξ denote the function defined for all $0 \leqq x \leqq 1$ by $G_\xi(x) \equiv G(x, \xi)$. Then,

(2.4) $D^{2n-1} G_\xi \in L^\infty[0, 1]$, and moreover $\sup\{\|D^{2n-1} G_\xi\|_\infty;\ 0 \leqq \xi \leqq 1\} = \sigma < +\infty$,

(2.5) $$D^k G_\xi(0) = D^k G_\xi(1) = 0 \quad \text{for all} \quad 0 \leqq k \leqq n-1,$$

and (as a consequence of (2.4) and (2.5))

(2.6) $$G_\xi \in S, \quad \text{and moreover} \quad \sup\{\|G_\xi\|_S;\ 0 \leqq \xi \leqq 1\} = \tau < +\infty,$$

(2.7) $$\lim_{x \to \xi^+} D^{2n-1} G_\xi(x) - \lim_{x \to \xi^-} D^{2n-1} G_\xi(x) = \frac{(-1)^n}{p_n(\xi)},$$

(2.8) $$L G_\xi(x) = 0 \quad \text{for all} \quad x \neq \xi.$$

Given any $f \in C^0[0, 1]$, the unique solution Φ of

(2.9) $$L u(x) = f(x) \quad \text{for all} \quad 0 < x < 1, \quad \text{and} \quad D^k u(0) = D^k u(1) = 0 \quad \text{for all} \quad 0 \leqq k \leqq n-1,$$

is given by

(2.10) $$\Phi(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi = (G_x, f).$$

The Green's function admits the absolutely and uniformly convergent expansion

(2.11) $$G(x, \xi) = \sum_{p=1}^{\infty} \frac{1}{\lambda_p} \Phi_p(x)\, \Phi_p(\xi) \quad \text{for all} \quad 0 \leqq x, \xi \leqq 1,$$

where λ_p, $p = 1, 2, \ldots$, are the eigenvalues of L arranged in increasing order of magnitude: $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$, and the functions Φ_p, $p = 1, 2, \ldots$, which belong to $C^{2n}[0, 1]$ and satisfy the boundary conditions of (1.4), are the associated eigenfunctions of L, i.e.,

(2.12) $$L\Phi_p = \lambda_p \Phi_p, \quad \text{or equivalently,} \quad a(\Phi_p, w) = \lambda_p(\Phi_p, w) \quad \text{for all} \quad w \in S, \quad p = 1, 2, \ldots,$$

these eigenfunctions being orthonormalized in the sense that

(2.13) $$(\Phi_p, \Phi_q) = \delta_{p,q}, \quad \text{and} \quad a(\Phi_p, \Phi_q) = \lambda_p \delta_{p,q},$$

where $\delta_{p,q}$ is the Kronecker delta.

Finally, given any $0 \leqq \xi \leqq 1$ and any $w \in S$, let

(2.14) $$F_\xi[w] \equiv a(w, w) - 2w(\xi).$$

Then, G_ξ strictly minimizes the functional $F_\xi[.]$ over the space S, i.e.,

(2.15) $$F_\xi[G_\xi] = \operatorname{Inf}\{F_\xi[w];\ w \in S\}.$$

Proof. With the assumptions of §1 for the operator L, the results of (2.1) to (2.13) are well-known (cf. Brauer [3], Kamke [13], and Courant-Hilbert [7, p. 360]). To prove (2.15), consider any $w \in S$, and write $w = \varepsilon + G_\xi$. Then,

$$F_\xi[w] = F_\xi[G_\xi] + a(\varepsilon, \varepsilon) + 2\{a(G_\xi, \varepsilon) - \varepsilon(\xi)\}.$$

But, using the result (2.26) of Theorem 3, to be proved later in this section, the term in brackets vanishes. Since $a(\varepsilon, \varepsilon) > 0$ if $\varepsilon \neq 0$, the conclusion of (2.15) follows. Q.E.D.

It is readily verified, using integration by parts, that the unique solution Φ of (2.9) satisfies

(2.16) $$a(\Phi, w) = (f, w) \quad \text{for all} \quad w \in S.$$

Given any finite-dimensional subspace S^N of S, the *variational approximation procedure*, as applied to the numerical solution of the boundary value problem (2.9), then consists in finding the unique element Φ^N in S^N which satisfies, in analogy with (2.16),

(2.17) $$a(\Phi^N, w) = (f, w) \quad \text{for all} \quad w \in S^N.$$

Remark. The function Φ^N may be equivalently characterized by the fact that $F[\Phi^N] = \operatorname{Inf}\{F[w];\ w \in S^N\}$, with $F[w] \equiv a(w, w) - 2(f, \omega)$. For details about the variational, or Rayleigh-Ritz procedures, see for example Mikhlin [15].

Following Ciarlet [5], we now characterize the unique element Φ^N in S^N which satisfies (2.17) in terms of a kernel $G^N(x, \xi)$. This is the object of the next theorem, which may be viewed as the discrete variational counterpart of Theorem 1. As the proof of Theorem 2 is straight-forward, we omit it.

Theorem 2. Given a finite dimensional subspace S^N of S, there exists a *discrete variational Green's function* $G^N(x, \xi)$ defined on the closed unit square $0 \leqq x, \xi \leqq 1$, which possesses the following properties:

(2.18) $$G^N(x, \xi) = G^N(\xi, x) \quad \text{in the closed unit square}.$$

Given any $0 \leqq \xi \leqq 1$, let G_ξ^N denote the function defined for all $0 \leqq x \leqq 1$ by $G_\xi^N(x) \equiv G^N(x, \xi)$. Then,

(2.19) $$G_\xi^N \in S \quad \text{for all} \quad 0 \leqq \xi \leqq 1, \quad \text{and} \quad \sup\{\|G_\xi^N\|_S;\ 0 \leqq \xi \leqq 1\} < \infty.$$

Given any $f \in C^0[0, 1]$, the unique variational approximation Φ^N in S^N, determined by (2.17), can be expressed as

$$\Phi^N(x) = \int_0^1 G^N(x, \xi) f(\xi) \, d\xi = (G_x^N, f). \tag{2.20}$$

The function $G^N(x, \xi)$ can be expanded as

$$G^N(x, \xi) = \sum_{p=1}^{N} \frac{1}{\lambda_p^N} \Phi_p^N(x) \Phi_p^N(\xi) \quad \text{for all} \quad 0 \leqq x, \xi \leqq 1, \tag{2.21}$$

where λ_p^N, $1 \leqq p \leqq N$, are the approximate eigenvalues of L arranged in increasing order of magnitude: $0 < \lambda_1^N \leqq \lambda_2^N \leqq \cdots \leqq \lambda_N^N$, and Φ_p^N, $1 \leqq p \leqq N$, are the orthonormalized approximate eigenfunctions of the operator L obtained by applying the Rayleigh-Ritz approximation procedure over the *same* subspace S^N, i.e., they are those functions of S^N which satisfy

$$a(\Phi_p^N, w) = \lambda_p^N (\Phi_p^N, w) \quad \text{for all} \quad w \in S^N, \quad 1 \leqq p \leqq N, \tag{2.22}$$

orthonormalized in the sense that

$$(\Phi_p^N, \Phi_q^N) = \delta_{p,q}, \quad \text{and} \quad a(\Phi_p^N, \Phi_q^N) = \lambda_p^N \delta_{p,q}. \tag{2.23}$$

For extensions of Theorem 2 to higher dimensions for second order operators, see [5].

We now come to an important property (Theorem 3) which is shared by both the Green's function and the discrete variational Green's function, but first we must recall some notions from functional analysis: Given a real Hilbert space H with inner product $(.,.)_H$ and whose elements are real-valued functions defined on a certain set X, the space H is said to have a *reproducing kernel* $K(x, \xi)$ if and only if the following two conditions are satisfied (cf. [1], [9, p. 316] or [21, p. 95]):

(2.24) For each $x \in X$, the function K_x is in H (again, K_x is the function defined for all $\xi \in X$ by $K_x(\xi) \equiv K(x, \xi)$).

(2.25) For each $x \in X$ and each $w \in H$, $w(x) = (K_x, w)_H$.

Then we have the following easily-established result (cf. [2] and [5]).

Theorem 3. The Green's function $G(x, \xi)$ is a reproducing kernel in the space S, i.e.,

$$w(x) = a(G_x, w) \quad \text{for all} \quad x \in [0, 1] \quad \text{and all} \quad w \in S, \tag{2.26}$$

and likewise, the discrete variational Green's function $G^N(x, \xi)$ is a reproducing kernel in the subspace S^N, i.e.,

$$w(x) = a(G_x^N, w) \quad \text{for all} \quad x \in [0, 1] \quad \text{and all} \quad w \in S^N. \tag{2.27}$$

Several consequences will now be derived from Theorem 3. The first one (Theorem 4) is the discrete version of the characterization of (2.15). Its importance lies in the fact that it gives a way of *actually computing* the function $G^N(x, \xi)$.

Theorem 4. For any fixed $\xi \in [0, 1]$, then G_ξ^N strictly minimizes the functional $F_\xi[w]$ of (2.14) over S^N, i.e.,

$$F_\xi[G_\xi^N] = \operatorname{Inf}\{F_\xi[w];\ w \in S^N\}. \tag{2.28}$$

Proof. For any $w \in S^N$, write $w = \varepsilon + G_\xi^N$. Then,

$$a(w, w) = a(G_\xi^N, G_\xi^N) + a(\varepsilon, \varepsilon) + 2a(G_\xi^N, \varepsilon) = a(G_\xi^N, G_\xi^N) + a(\varepsilon, \varepsilon) + 2\varepsilon(\xi),$$

by making use of (2.27). Thus, $F_\xi[w] = F_\xi[G_\xi^N] + a(\varepsilon, \varepsilon)$, which completes the proof. Q.E.D.

An essential feature possessed by the variational approximation procedure is that (cf. [20]) the approximation Φ^N is the *projection* of the solution Φ on the subspace S^N in the sense of the inner product $a(.,.)$, i.e.,

$$a(\Phi - \Phi^N, w) = 0 \quad \text{for all} \quad w \in S^N. \tag{2.29}$$

We now prove that a similar property holds for the discrete variational Green's function:

Theorem 5. For any fixed $\xi \in [0, 1]$, then

$$a(G_\xi - G_\xi^N, w) = 0 \quad \text{for all} \quad w \in S^N. \tag{2.30}$$

Proof. It suffices to apply both relations (2.26) and (2.27) to any function $w \in S^N$. Q.E.D.

As an immediate consequence of Theorem 5, we now have

Corollary 1. For any fixed $\xi \in [0, 1]$, then

$$\|G_\xi - G_\xi^N\|_S = \inf\{\|G_\xi - w\|_S;\ w \in S^N\}, \tag{2.31}$$

and

$$\|D^k(G_\xi - G_\xi^N)\| \leqq A_k \inf\{\|G_\xi - w\|_S;\ w \in S^N\}, \qquad \text{for all} \quad 0 \leqq k \leqq n, \tag{2.32}$$

$$\|D^l(G_\xi - G_\xi^N)\|_\infty \leqq B_l \inf\{\|G_\xi - w\|_S;\ w \in S^N\}, \qquad \text{for all} \quad 0 \leqq l \leqq n-1, \tag{2.33}$$

where A_k and B_l are positive constants.

Proof. The relation (2.31) follows directly from (2.30), and the inequalities of (2.32) and (2.33) are mere applications of (1.6), (1.7), and (1.10). Q.E.D.

The quantity $\inf\{\|G_\xi - w\|_S;\ w \in S^N\}$ which appears in the error bound (2.31) of Corollary 1 depends upon ξ, and it is of course desirable to have an error bound valid for all $\xi \in [0, 1]$. This is given in Corollary 2 below. To this end, we define S_∞ as the space whose elements u are functions $u: (x, \xi) \to u(x, \xi)$ defined on the unit square $0 \leqq x, \xi \leqq 1$ such that

i) if $u_\xi(x) \equiv u(x, \xi)$, then $u_\xi \in S$ for all $0 \leqq \xi \leqq 1$,

ii) $\sup\{\|u_\xi\|_S;\ 0 \leqq \xi \leqq 1\} < \infty$.

Then, S_∞ is a normed linear space with respect to the norm defined by

$$|||u|||_\infty \equiv \sup\{\|u_\xi\|_S;\ 0 \leqq \xi \leqq 1\}. \tag{2.34}$$

From Theorems 1 and 2, we see that $G(x, \xi)$ and $G^N(x, \xi)$ are elements of S_∞.

Corollary 2. If we identify a function w: $x\in[0, 1]\to w(x)$ of S^N with the function w: $(x, \xi)\to w(x)$ of S_∞ for all $0\leqq\xi\leqq 1$, then

$$\text{(2.35)}\qquad |||G-G^N|||_\infty \leqq \inf\{|||G-w|||_\infty;\ w\in S^N\}.$$

Proof. For any $w\in S^N$, we have from (2.31) that $\|G_\xi-G_\xi^N\|_S\leqq\|G_\xi-w\|_S$ for all $0\leqq\xi\leqq 1$. Hence, from the monotonic character of the L^∞-norm (i.e., if $0\leqq U(\xi)\leqq V(\xi)$ for all $0\leqq\xi\leqq 1$, then $\|U\|_\infty\leqq\|V\|_\infty$), it follows that $|||G-G^N|||_\infty\leqq |||G-w|||_\infty$, from which (2.35) follows. Q.E.D.

As a consequence we now have a *convergence criterion*:

Theorem 6. Let $\{S^{N_i}\}_{i=1}^\infty$ be a sequence of (not necessarily nested) subspaces of S. If for a fixed $\xi\in[0, 1]$,

$$\text{(2.36)}\qquad \lim_{i\to+\infty}[\inf\{\|G_\xi-w\|_S;\ w\in S^{N_i}\}]=0,$$

then

$$\text{(2.37)}\qquad \lim_{i\to+\infty}\|G_\xi-G_\xi^{N_i}\|_S=0.$$

Simearly, if

$$\text{(2.38)}\qquad \lim_{i\to+\infty}[\inf\{|||G-w|||_\infty;\ w\in S^{N_i}\}]=0,$$

then

$$\text{(2.39)}\qquad \lim_{i\to+\infty}|||G-G^{N_i}|||_\infty=0.$$

Proof. The first part of Theorem 6 is a direct consequence of Corollary 1, and likewise, the second part of Theorem 6 is a direct consequence of Corollary 2. Q.E.D.

Remark. If condition (2.38) of Theorem 6 holds, then it follows by using (1.7) with $l=0$ that the sequence $\{G^{N_i}(x, \xi)\}_{i=1}^\infty$ converges *uniformly* to $G(x, \xi)$ in the closed unit square, i.e.,

$$\lim_{i\to+\infty}[\sup\{|G^{N_i}(x, \xi)-G(x, \xi)|;\ 0\leqq x, \xi\leqq 1\}]=0.$$

In the next sections, we will prove directly similar results in particular subspaces of S.

§ 3. The Eigensubspace Σ^N

As in § 2, let λ_p and Φ_p, $p=1, 2, \ldots$, denote respectively the eigenvalues and the associated eigenfunctions of L, orthonormalized as in (2.13).

The subspace of S spanned by the first N eigenfunctions Φ_p, $1\leqq p\leqq N$, will be called the *eigensubspace* Σ^N. In this particular case, the corresponding discrete variational Green's function is, from (2.21), given by (cf. [5, Corollary 1])

$$\text{(3.1)}\qquad G^N(x, \xi)=\sum_{p=1}^N \frac{1}{\lambda_p}\Phi_p(x)\,\Phi_p(\xi)\qquad \text{for all}\quad 0\leqq x, \xi\leqq 1,$$

and thus (by (2.11))

$$\text{(3.2)}\qquad G(x, \xi)-G^N(x, \xi)=\sum_{p=N+1}^\infty \frac{1}{\lambda_p}\Phi_p(x)\,\Phi_p(\xi)\quad \text{for all}\quad 0\leqq x, \xi\leqq 1.$$

As a consequence, some of the results proved in § 2 (such as Theorem 5 for example) become obvious; but, in addition, the relations (3.1) and (3.2) allow us to give more precise answers to other questions, such as how well the function $G^N(x, \xi)$ approximates the Green's function $G(x, \xi)$. We begin with

Theorem 7. Let the operator L of (1.1) be the second-order operator

$$L u(x) = -D\{p_1(x) D u(x)\} + p_0(x) u(x).$$

Then, the associated discrete variational Green's function $G^N(x, \xi)$ obtained over the subspace Σ^N is such that

$$\text{(3.3)} \qquad \operatorname{Sup}\{|G(x, \xi) - G^N(x, \xi)|;\ 0 \leqq x, \xi \leqq 1\} \leqq \frac{K}{N} \quad \text{for all} \quad N \geqq 1,$$

where K is a constant independent of N. Moreover, the above asymptotic order of convergence is best possible.

Proof. By the assumptions of (1.2) and (1.3), p_1 is of class $C^2[0, 1]$ and is strictly positive on $[0, 1]$, so that we may reduce the operator L to a simpler form by using a classical change of dependent and independent variables (cf. [7, p. 336]). Specifically, let

$$v(x) = \{p_1(x)\}^{\frac{1}{4}} u(x), \quad \text{and} \quad t = \frac{1}{K}\int_0^x \{p_1(\eta)\}^{-\frac{1}{2}} d\eta, \quad \text{with} \quad K = \int_0^1 \{p_1(\eta)\}^{-\frac{1}{2}} d\eta.$$

Then, $\{\lambda_p, \Phi_p(x)\}$ is an eigensolution of

$$L u(x) = \lambda u(x), \quad 0 < x < 1, \quad \text{with} \quad u(0) = u(1) = 0,$$

if and only if $\{\mu_p = K^2 \lambda_p,\ \Psi_p(t) = \{p_1(x)\}^{\frac{1}{4}} \Phi_p(x)\}$ is an eigensolution of

$$-\frac{d^2 v}{dt^2}(t) + \varrho(t) v(t) = \mu v(t), \quad 0 < t < 1, \quad \text{with} \quad v(0) = v(1) = 0.$$

Here, x is understood as being a function of t. Note that

$$\varrho(t) = K^2 p_0(x) - \{p_1(x)\}^{-\frac{1}{4}} \frac{d}{dt}\left(\{p_1(x)\}^{-\frac{1}{4}}\right)$$

is, by virtue of the assumptions upon p_0, p_1, a continuous function for $0 \leqq t \leqq 1$. Thus, we can apply to this simpler problem the known asymptotic formulas [7, p. 336]:

$$\mu_p = p^2 \pi^2 + O(1), \quad \text{and} \quad \Psi_p(t) = \sqrt{2} \sin p \pi t + O\left(\frac{1}{p}\right) \quad \text{as} \quad p \to +\infty,$$

the eigenfunctions Ψ_p being orthonormalized in the L^2-sense. Thus, going back to the eigenvalues and eigenfunctions of the operator L, we see that

$$\lambda_p = \frac{\pi^2}{K^2} p^2 + O(1), \quad \text{and} \quad \|\Phi_p\|_\infty = O(1) \quad \text{as} \quad p \to +\infty,$$

so that we obtain, by making use of (3.2),

$$|G(x, \xi) - G^N(x, \xi)| = O\left(\sum_{p=N+1}^{\infty} \frac{1}{p^2}\right) = O\left(\frac{1}{N}\right)$$

uniformly in the unit square $0 \leqq x, \xi \leqq 1$, which proves (3.3).

To show that the asymptotic order of convergence is best possible, consider the special case where $L=-D^2$. Then, $\lambda_p = p^2\pi^2$, $\Phi_p(x)=\sqrt{2}\sin p\pi x$, $p=1, 2, \ldots,$ and

$$G\left(\frac{1}{2}, \frac{1}{2}\right) - G^N\left(\frac{1}{2}, \frac{1}{2}\right) = \sum_{\substack{p=N+1 \\ p \text{ odd}}}^{\infty} \frac{1}{p^2\pi^2} = O\left(\frac{1}{N}\right). \quad \text{Q.E.D.}$$

Extending the classical definition for second-order operators [8, p. 326], we say that a $2n$-th order differential operator L such as that of (1.1) satisfies the maximum principle if any function $\Phi \in C^{2n}[0, 1]$ which satisfies

$$L\Phi(x) = f(x) \geqq 0, \quad 0<x<1, \quad \text{and} \quad D^k\Phi(0) = D^k\Phi(1) = 0, \quad 0 \leqq k \leqq n-1,$$

is such that

$$\Phi(x) \geqq 0, \quad 0 \leqq x \leqq 1.$$

By the representation (2.10) of Theorem 1, it is clear that the maximum principle holds if and only if

$$G(x, \xi) \geqq 0 \quad \text{in} \quad 0 \leqq x, \xi \leqq 1.$$

Likewise, if Φ^N is the variational approximation to the solution of (2.9), we say [5] that a *discrete variational maximum principle* holds in the given finite dimensional subspace S^N of S if and only if

$$\Phi^N(x) \geqq 0, \quad 0 \leqq x \leqq 1, \quad \text{whenever} \quad f(x) \geqq 0, \quad 0 \leqq x \leqq 1.$$

Equivalently, by the representation (2.20) of Theorem 2, the above discrete variational maximum principle is valid if and only if

$$G^N(x, \xi) \geqq 0 \quad \text{in} \quad 0 \leqq x, \xi \leqq 1.$$

We now prove

Theorem 8. In the special case where $L=-D^2$, the discrete variational maximum principle holds in the eigensubspace Σ^N for any $N \geqq 1$.

Proof. We must prove that $G^N(x, \xi) \geqq 0$ in the closed unit square $0 \leqq x, \xi \leqq 1$, with

$$G^N(x, \xi) = \frac{2}{\pi^2} \sum_{p=1}^{N} \frac{1}{p^2} \sin p\pi x \sin p\pi\xi.$$

Since $G^N(x, \xi) = G^N(\xi, x)$ and $G^N(x, \xi) = G^N(1-\xi, 1-x)$, we can restrict ourselves to the portion $0 \leqq x-\xi \leqq x+\xi \leqq 1$ of the unit square, where we have

$$\begin{aligned} G^N(x, \xi) &= \sum_{p=1}^{N} \frac{1}{\pi^2 p^2} \{\cos p\pi(x-\xi) - \cos p\pi(x+\xi)\} \\ &= \int_{x-\xi}^{x+\xi} \left\{ \sum_{p=1}^{N} \frac{\sin p\pi t}{p\pi} \right\} dt, \end{aligned}$$

so that $G^N(x, \xi)$ is a nonnegative function if and only if

$$s_N(\theta) \equiv \sum_{p=1}^{N} \frac{\sin p\,\theta}{p} \geqq 0 \quad \text{for all} \quad 0 \leqq \theta \leqq \pi.$$

To prove this, observe that $s_1(\theta) \geqq 0$ for all $0 \leqq \theta \leqq \pi$, and assume that the property is true up to $s_{N-1}(\theta)$. The relative minima of the function $s_N(\theta)$ over $[0, \pi]$ occur [17, p. 78] at the points $\theta_l = \frac{2\pi l}{N}$ for $l = 1, 2, \ldots, \left[\frac{N-1}{2}\right]$, where $[\alpha]$ denotes the integer part of the real number α. At such a point θ_l, $s_N(\theta_l) = s_{N-1}(\theta_l)$ since $\sin N\theta_l = 0$, and as a consequence of the induction hypothesis, i.e., that $s_{N-1}(\theta) \geqq 0$ for all $0 \leqq \theta \leqq \pi$, it follows that $s_N(\theta_l) \geqq 0$ for all $l = 1, 2, \ldots, \left[\frac{N-1}{2}\right]$, which completes the proof. Q.E.D.

§ 4. The L-Spline Subspace $\mathrm{Sp}(L, \Pi^N, z)$

Given the operator L of (1.1), and given any (not necessarily uniform) partition Π^N: $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ of the unit interval, let $z = (z_1, z_2, \ldots, z_N)$, the *incidence vector* associated with Π^N, be an N-vector with positive integer components satisfying $1 \leqq z_i \leqq n$ for all $1 \leqq i \leqq N$. Then, any real-valued function defined on $[0, 1]$ is said to be an *L-spline* if

$$\begin{aligned} & s \in C^{2n}[x_i, x_{i+1}] \quad \text{for each } i, \quad 0 \leqq i \leqq N, \\ (4.1) \qquad & L s(x) = 0 \quad \text{for all} \quad x \in (x_i, x_{i+1}) \quad \text{for each } i, \quad 0 \leqq i \leqq N, \quad \text{and} \\ & D^k s(x_i -) = D^k s(x_i +) \quad \text{for} \quad 0 \leqq k \leqq 2n - 1 - z_i, \quad 1 \leqq i \leqq N. \end{aligned}$$

The class of all L-splines for fixed Π^N and z is denoted by $\mathrm{Sp}(L, \Pi^N, z)$. Such spaces, recently studied by Lucas [14] and Golomb and Jerome [10], are extensions of the L-splines studied in [19], in that it was assumed in [19] that L has the special form $L = -\mathscr{L}^* \mathscr{L}$, where $\mathscr{L}$ is an n-th order linear differential operator and $\mathscr{L}^*$ is its formal adjoint.

For our purposes here, we shall restrict our attention to L-spline subspaces of S, i.e., we consider only those elements of $\mathrm{Sp}(L, \Pi^N, z)$ which satisfy the homogeneous boundary conditions of (1.4). We shall denote such subspaces of S by $\mathrm{Sp}_0(L, \Pi^N, z)$. It is readily verified that the dimension of $\mathrm{Sp}_0(L, \Pi^N, z)$ is $\sum_{i=1}^{N} z_i$.

We now give a result of Lucas [14]. This type of result can be traced back to Rose [18], and many authors (cf. [4, 12, 16]) have contributed generalizations of Rose's results.

Theorem 9. For a given $f \in C^0[0, 1]$, let Φ denote the unique solution of the boundary value problem (2.9), and let Φ^N denote the approximation of Φ in $\mathrm{Sp}_0(L, \Pi^N, z)$ which satisfies (2.17). For convenience, we define $z_0 = z_{N+1} = n$. Then,

$$(4.2) \qquad D^k \Phi^N(x_i) = D^k \Phi(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 0 \leqq i \leqq N + 1.$$

Again, a property similar to that established in (4.2) holds for the function $G^N(x, \xi)$. This is the object of

Theorem 10. Let $G^N(x, \xi)$ be the discrete variational Green's function associated with the L-spline subspace $\mathrm{Sp}_0(L, \Pi^N, z)$. Then, for any $\xi \in [0, 1]$,

(4.3) $$D^k G_\xi^N(x_i) = D^k G_\xi(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 0 \leqq i \leqq N+1.$$

Proof. From the representations of (2.10) and (2.20), we have for any $1 \leqq i \leqq N$,

$$D^k \Phi(x_i) = \int_0^1 \frac{\partial^k G}{\partial x_k}(x_i, \xi) f(\xi)\, d\xi, \qquad 0 \leqq k \leqq z_i - 1,$$

and

$$D^k \Phi^N(x_i) = \int_0^1 \frac{\partial^k G^N}{\partial x^k}(x_i, \xi) f(\xi)\, d\xi, \qquad 0 \leqq k \leqq z_i - 1.$$

The differentiation under the integral can be indeed carried out for both $G(x, \xi)$ and $G^N(x, \xi)$ since $\frac{\partial^k G}{\partial x^k}(x, \xi)$ as well as $\frac{\partial^k G^N}{\partial x^k}(x, \xi)$ are continuous functions in the closed unit square for all $0 \leqq k \leqq n-1$ (cf. Theorems 1 and 2).

Thus, by applying Theorem 9, we must have, for any $1 \leqq i \leqq N$,

$$\int_0^1 \left\{ \frac{\partial^k G}{\partial x^k}(x_i, \xi) - \frac{\partial^k G^N}{\partial x^k}(x_i, \xi) \right\} f(\xi)\, d\xi = 0, \qquad 0 \leqq k \leqq z_i - 1,$$

and the relations (4.3) are then a consequence of the fact that the above relation holds for any function $f \in C^0[0, 1]$. Q.E.D.

Given a partition Π^N of the unit interval, let D_{ij} denote the open subsets of the unit square defined by

$$D_{ij} = \{(x, \xi) \in R^2;\ x_i < x < x_{i+1},\ x_j < \xi < x_{j+1}\}, \qquad 0 \leqq i, j \leqq N.$$

Theorem 11 (*The "Diagonal Squares Theorem"*). Let $G^N(x, \xi)$ denote the discrete variational Green's function for the *Hermite L-spline subspace* $\mathrm{Sp}_0(L, \Pi^N, \hat{z})$, where $\hat{z} \equiv (n, n, \ldots, n)$. Then,

(4.4) $$G^N(x, \xi) = G(x, \xi) \quad \text{except for} \quad (x, \xi) \in D_{ii}, \quad 0 \leqq i \leqq N.$$

In other words, $G^N(x, \xi)$ coincides with $G(x, \xi)$ over the closed unit square except over the open diagonal squares formed by the partition Π^N (cf. Fig. 1).

Proof. Let $\xi \in [0, 1]$ be given with $\xi \in (x_{i_0}, x_{i_0+1})$ for some $0 \leqq i_0 \leqq N$. Then, for any $0 \leqq i \leqq N$, we have, by Theorem 10,

$$D^k(G_\xi(x_i) - G_\xi^N(x_i)) = D^k((G_\xi(x_{i+1}) - G_\xi^N(x_{i+1}))) = 0, \qquad 0 \leqq k \leqq n-1,$$

and, if moreover $i \neq i_0$,

$$L(G_\xi(x) - G_\xi^N(x)) = 0 \quad \text{for} \quad x_i < x < x_{i+1},$$

since on the one hand $LG_\xi(x) = 0$ (cf. (2.8)) and on the other hand $LG_\xi^N(x) = 0$ in (x_i, x_{i+1}) (cf. (4.1)). Thus, it follows that $G_\xi(x) \equiv G_\xi^N(x)$ for $x_i < x < x_{i+1}$, $i \neq i_0$, by virtue of the uniqueness of the solution of the boundary value problem

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Corrections

#59

P, 121, line +14. Read "Similarly"

P. 127, line +13-14. Read "positive constants"