SIAM J. NUMER. ANAL.
Vol. 7, No. 4, December 1970

PATTERNS OF DEPENDENCE IN GENERALIZATIONS OF GERSCHGORIN'S THEOREM*

ALAN J. HOFFMAN† AND RICHARD S. VARGA‡

This paper is dedicated to Professor Alston S. Householder on the occasion of his 65th birthday.

1. Introduction. A well-known theorem of Gerschgorin asserts that every eigenvalue of an $n \times n$ complex matrix $A = (a_{i,j})$ lies in the union of the following n disks in the complex plane:

$$d_k \equiv \left\{ z : |a_{k,k} - z| \leqq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{k,j}| \right\}, \qquad k = 1, 2, \cdots, n. \tag{1.1}$$

Our interest here concerns a problem associated with the following type of generalization of Gerschgorin's theorem, in which the radii of the disks of (1.1) are nonnegative functions, denoted by $f_k(A)$, defined on the moduli of the $n(n-1)$ off-diagonal entries $a_{i,j}$ of the matrix A. We say that a set $\{f_1, f_2, \cdots, f_n\}$ of such functions is a *G-generating family* if and only if for every $n \times n$ complex matrix $A = (a_{i,j})$, every eigenvalue of A lies in the union of the n disks

$$d_k = \{z : |a_{k,k} - z| \leqq f_k(A)\}, \qquad k = 1, 2, \cdots, n. \tag{1.2}$$

Equivalently, the set $\{f_1, f_2, \cdots, f_n\}$ is a G-generating family if and only if for every $n \times n$ complex matrix $A = (a_{i,j})$ satisfying

$$|a_{k,k}| > f_k(A), \qquad k = 1, 2, \cdots, n, \tag{1.3}$$

A is nonsingular. For example, $\{f_k(A) \equiv \sum_{j=1, j\neq k}^{n} |a_{k,j}|\}_{k=1}^{n}$ is obviously a G-generating family, and various G-generating families are easily constructed from the many known generalizations of the Gerschgorin theorem. The concept of G-generating families seems first to have appeared in Nowosad [2]. For theoretical results concerning properties of G-generating families and their connections with classical Gerschgorin-type theorems, see Hoffman [1].

Our problem here treats the question of *patterns of dependence* of the functions f_k on the variables $|a_{i,j}|$, $i \neq j$. We first say that f_k *depends* on the ordered pair of positive integers (i, j), where $1 \leqq i, j \leqq n$ with $i \neq j$, if and only if there exist $n \times n$ complex matrices $A = (a_{k,l})$ and $B = (b_{k,l})$ such that $|a_{k,l}| = |b_{k,l}|$ for all $k \neq l$ with $(k, l) \neq (i, j)$, for which $f_k(A) \neq f_k(B)$. We then define

$$D(f_k) = \{(i, j) : 1 \leqq i, j \leqq n \text{ and } f_k \text{ depends on } (i, j)\} \tag{1.4}$$

as the *domain of dependence of* f_k.

* Received by the editors July 7, 1970.

† IBM Research Center, Yorktown Heights, New York 10598. The work of this author was supported in part by the Office of Naval Research under Contract Nonr-3775(00).

‡ Department of Mathematics, Kent State University, Kent, Ohio 44240. The work of this author was supported in part by the Atomic Energy Commission under AEC Grant AT(11-1)-2075.

It is very natural to pose the following problem. If $D_1, D_2, \cdots, D_n$ are arbitrary subsets of the ordered pairs of positive integers (i, j), where $1 \leqq i, j \leqq n$ with $i \neq j$, what are necessary and sufficient conditions for the existence of a G-generating family $\{f_1, f_2, \cdots, f_n\}$ with

$$D_k = D(f_k), \qquad k = 1, 2, \cdots, n? \tag{1.5}$$

The main purpose of the note is to solve this problem (Theorem 1), under the assumptions that the functions f_k, $k = 1, 2, \cdots, n$, are *homogeneous* (of degree unity); i.e., for every $\varepsilon > 0$ and every $n \times n$ matrix A,

$$f_k(\varepsilon A) = \varepsilon f_k(A), \qquad k = 1, 2, \cdots, n, \tag{1.6}$$

and that the functions f_k are *bounded on bounded sets*; i.e., for all $n \times n$ matrices $A = (a_{i,j})$ with $|a_{i,j}| \leqq c$ for all $1 \leqq i, j \leqq n$ with $i \neq j$, there exist positive constants $M_k(c)$ such that

$$f_k(A) \leqq M_k(c), \qquad k = 1, 2, \cdots, n. \tag{1.7}$$

The assumption of (1.7) is surely satisfied if each f_k is continuous in its $n(n-1)$ arguments $|a_{1,2}|, \cdots, |a_{n,n-1}|$. Note that assumptions of (1.6) and (1.7) are trivially satisfied for the G-generating family $\{f_k(A) = \sum_{j=1, j\neq k}^{n} |a_{k,j}|\}_{k=1}^{n}$ of (1.1).

In the subsequent material, for any subset $Q \subset \{1, 2, \cdots, n\}$, we use the notation $|Q|$ to denote the number of elements of Q.

2. Main result. Our main result is the following theorem.

THEOREM 1. *Let $D_1, D_2, \cdots, D_n$ be subsets of the set of all ordered pairs (i, j), where $1 \leqq i, j \leqq n$ with $i \neq j$. Then, there exists a G-generating family $\{f_1, f_2, \cdots, f_n\}$ with each f_k homogeneous* (*cf.* (1.6)) *and bounded on bounded sets* (*cf.* (1.7)) *satisfying*

$$D_k = D(f_k), \qquad k = 1, 2, \cdots, n, \tag{2.1}$$

if and only if for every subset $S \subset \{1, 2, \cdots, n\}$ with $|S| \geqq 2$, for every cyclic permutation σ of S, and for every nonempty subset $T \subset S$,

$$\left| \left\{ i : i \in S \text{ and } \{(i, \sigma i)\} \in \bigcup_{k \in T} D_k \right\} \right| \geqq |T|. \tag{2.2}$$

Proof. Assuming (2.1), we first show the necessity of (2.2). Assume the contrary. Then, there exists an $S \subset \{1, 2, \cdots, n\}$, with $|S| \geqq 2$, a cyclic permutation σ of S, a nonempty subset $T \subset S$, and indices $i_1, \cdots, i_r \in S$ such that

$$\bigcup_{j=1}^{r} \{(i_j, \sigma i_j)\} = \bigcup_{i \in S} \{(i, \sigma i)\} \cap \bigcup_{k \in T} D(f_k), \tag{2.3}$$

with $r < t \equiv |T|$. Next, given any ε with $0 < \varepsilon < 1$, define the off-diagonal entries of the $n \times n$ matrix $A(\varepsilon) = (a_{i,j}(\varepsilon))$ by

$$\begin{aligned} a_{i_j, \sigma i_j} &= -\varepsilon, \quad j = 1, 2, \cdots, r, \\ a_{i, \sigma i} &= -1, \quad i \in S, \quad i \neq i_1, i_2, \cdots, i_r, \\ a_{k,l} &= 0, \quad \text{otherwise } (k \neq l). \end{aligned}$$

Note that any ordered pair $(i, \sigma i)$, $i \in S$, $i \neq i_1, i_2, \cdots, i_r$, is, from (2.3), *not* in any $D(f_k)$, $k \in T$. In other words, for any $k \in T$, $f_k(A(\varepsilon))$ depends only on coefficients of modulus ε or zero. Hence, by virtue of the assumption that each f_k is homogeneous, it follows (cf. (1.6)) that

$$f_k(A(\varepsilon)) = \varepsilon f_k(A(1)) \quad \text{for all } k \in T. \tag{2.4}$$

Next, consider $g(\varepsilon) \equiv \prod_{k \in S} f_k(A(\varepsilon))$. This can also be written as

$$g(\varepsilon) = \prod_{k \in T} f_k(A(\varepsilon)) \prod_{k \in S - T} f_k(A(\varepsilon)) = \varepsilon^t \prod_{k \in T} f_k(A(1)) \prod_{k \in S - T} f_k(A(\varepsilon)),$$

using the expression of (2.4). Because the off-diagonal entries of $A(\varepsilon)$ are in modulus at most unity, it follows from the boundedness assumption on the f_k that (cf. 1.7))

$$\prod_{k \in S - T} M_k(1) \geqq \prod_{k \in S - T} f_k(A(\varepsilon)).$$

Thus, as $t > r$, we see that for all sufficiently small $\varepsilon > 0$,

$$\varepsilon^r > \varepsilon^t \left\{ \prod_{k \in T} f_k(A(1)) \prod_{k \in S - T} M_k(1) \right\} \geqq g(\varepsilon).$$

Fixing ε sufficiently small, this means that we can find positive numbers $c_k = c_k(\varepsilon)$, $k \in S$, with

$$c_k > f_k(A(\varepsilon)), \qquad k \in S,$$

and

$$\varepsilon^r = \prod_{k \in S} c_k > g(\varepsilon).$$

Now, we define the diagonal entries of $A(\varepsilon)$ as

$$a_{j,j}(\varepsilon) = c_j, \qquad j \in S,$$
$$a_{j,j}(\varepsilon) > f_j(A(\varepsilon)), \qquad j \notin S.$$

By means of our construction, we evidently have that $a_{j,j} > f_j(A(\varepsilon))$ for all $j = 1, 2, \cdots, n$. But, as $\{f_1, f_2, \cdots, f_n\}$ is by assumption a G-generating family, it follows from (1.3) that $A(\varepsilon)$ is nonsingular. On the other hand, our construction gives us that $A(\varepsilon)$ is (after a suitable permutation) the direct sum of an $|S| \times |S|$ submatrix and a $(n - |S|) \times (n - |S|)$ positive diagonal matrix. This $|S| \times |S|$ submatrix, however, has determinant zero; the product of its diagonal entries is ε^r, and the only other nonzero term contributing to its determinant is $\operatorname{sgn} \sigma \prod_{i \in S} a_{i,\sigma i} = -\varepsilon^r$. Thus, $A(\varepsilon)$ is singular, a contradiction which establishes the validity of (2.2).

Conversely, assume (2.2), and define the functions f_k by means of

$$f_k(A) = \sqrt{n!} \sum_{(i,j) \in D_k} |a_{i,j}|, \qquad k = 1, 2, \cdots, n. \tag{2.5}$$

These functions f_k satisfy (2.1) and are obviously homogeneous and bounded on bounded sets.

It remains to show that $\{f_1, f_2, \cdots, f_n\}$ is a G-generating family. From (1.3), it is sufficient to show that for any $n \times n$ matrix $A = (a_{i,j})$ with $|a_{k,k}| > f_k(A)$ for $k = 1, 2, \cdots, n$, A is nonsingular. Thus, we assume that

$$|a_{k,k}| > f_k(A), \qquad k = 1, 2, \cdots, n. \tag{2.6}$$

Consider any subset $S \subset \{1, 2, \cdots, n\}$ with $|S| \geqq 2$ and any cyclic permutation σ on S, and define the $|S| \times |S|$ matrix $B = (b_{i,k})$, $i, k \in S$, by

$$b_{i,k} = \begin{cases} 1 & \text{if } (i, \sigma i) \in D_k, \\ 0 & \text{if } (i, \sigma i) \notin D_k. \end{cases}$$

Noting that B is a zero-one matrix, the assumption of (2.2) is precisely the condition given by Philip Hall in his famous theorem on systems of distinct representatives (cf. Ryser [3]) that there exists a permutation τ on S with $b_{i,\tau i} = 1$ for all $i \in S$. Thus, by definition, $(i, \sigma i) \in D_{\tau i}$ for all $i \in S$; and from (2.5), we deduce that $f_{\tau i}(A) \geqq \sqrt{n!}|a_{i,\sigma i}|$ for all $i \in S$. Consequently, from (2.6),

$$|a_{\tau i,\tau i}| > f_{\tau i}(A) \geqq \sqrt{n!}|a_{i,\sigma i}|, \qquad i \in S.$$

Taking products in the above expression over all $i \in S$, and noting that $|S| \geqq 2$, we have

$$\prod_{i \in S} |a_{i,i}| > (n!)^{|S|/2} \prod_{i \in S} |a_{i,\sigma i}| \geqq (n!) \prod_{i \in S} |a_{i,\sigma i}| \tag{2.7}$$

for *any* subset $S \subset \{1, 2, \cdots, n\}$ with $|S| \geqq 2$ and *any* cyclic permutation σ on S. Recalling that the determinant of A is the sum of $n!$ products of elements of A, it is easy to see that (2.7) implies that A is nonsingular. This completes the proof.

We remark that with the assumption of (2.2), it can be shown by more intricate arguments that the functions f_k of (2.5), even with the factor $\sqrt{n!}$ deleted, still form a G-generating family (cf. Hoffman [1]).

REFERENCES

[1] A. J. HOFFMAN, *Generalizations of Gersgorin's theorem: G-generating families*, Lecture notes, University of California at Santa Barbara, 1969, 46 pp.

[2] P. NOWOSAD, *On the functional (x^{-1}, Ax) and some of its applications*, An. Acad. Brasil Ci., 37 (1965), pp. 163–165.

[3] H. J. RYSER, *Combinatorial Mathematics*, Carus Mathematical Monograph No. 14, John Wiley, New York, 1963.