PATTERNS OF DEPENDENCE IN GENERALIZATIONS OF GERSHGORIN'S THEOREM

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This paper is dedicated to Professor Alston S. Householder on the occasion of his 65th birthday.

1. Introduction. A well-known theorem of Gerschgorin asserts that every eigenvalue of an \( n \times n \) complex matrix \( A = (a_{ij}) \) lies in the union of the following \( n \) disks in the complex plane:

\[
d_k = \left\{ z : |a_{kk} - z| \leq \sum_{j \neq k} |a_{kj}| \right\}, \quad k = 1, 2, \ldots, n.
\]

Our interest here concerns a problem associated with the following type of generalization of Gerschgorin's theorem, in which the radii of the disks of (1.1) are nonnegative functions, denoted by \( f_k(A) \), defined on the moduli of the \( n(n-1) \) off-diagonal entries \( a_{ij} \) of the matrix \( A \). We say that a set \( \{f_1, f_2, \ldots, f_n\} \) of such functions is a G-generating family if and only if for every \( n \times n \) complex matrix \( A = (a_{ij}) \), every eigenvalue of \( A \) lies in the union of the \( n \) disks

\[
d_k = \left\{ z : |a_{kk} - z| \leq f_k(A) \right\}, \quad k = 1, 2, \ldots, n.
\]

Equivalently, the set \( \{f_1, f_2, \ldots, f_n\} \) is a G-generating family if and only if for every \( n \times n \) complex matrix \( A = (a_{ij}) \) satisfying

\[
|a_{kk}| > f_k(A), \quad k = 1, 2, \ldots, n,
\]

\( A \) is nonsingular. For example, \( \{f_k(A) = \sum_{j=1,j \neq k}^{n} |a_{kj}| \} \) is obviously a G-generating family, and various G-generating families are easily constructed from the many known generalizations of the Gershgorin theorem. The concept of G-generating families seems first to have appeared in Nowosad [2]. For theoretical results concerning properties of G-generating families and their connections with classical Gershgorin-type theorems, see Hoffman [1].

Our problem here treats the question of patterns of dependence of the functions \( f_k \) on the variables \( |a_{ij}| \), \( i \neq j \). We first say that \( f_k \) depends on the ordered pair of positive integers \( (i, j) \), where \( 1 \leq i, j \leq n \) with \( i \neq j \), if and only if there exist \( n \times n \) complex matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) such that \( |a_{kk}| = |b_{kk}| \) for all \( k \neq l \) with \( (k, l) \neq (i, j) \), for which \( f_k(A) \neq f_k(B) \). We then define

\[
D(f_k) = \{(i, j) : 1 \leq i, j \leq n \text{ and } f_k \text{ depends on } (i, j)\}
\]

as the domain of dependence of \( f_k \).

\* Received by the editors July 7, 1970.
\* IBM Research Center, Yorktown Heights, New York 10598. The work of this author was supported in part by the Office of Naval Research under Contract Nonr-3775(00).
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It is very natural to pose the following problem. If $D_1, D_2, \cdots, D_n$ are arbitrary subsets of the ordered pairs of positive integers $(i, j)$, where $1 \leq i, j \leq n$ with $i \neq j$, what are necessary and sufficient conditions for the existence of a $G$-generating family $\{f_1, f_2, \cdots, f_n\}$ with

$$D_k = D(f_k), \quad k = 1, 2, \cdots, n?$$

The main purpose of the note is to solve this problem (Theorem 1), under the assumptions that the functions $f_k$, $k = 1, 2, \cdots, n$, are homogeneous (of degree unity); i.e., for every $c > 0$ and every $n \times n$ matrix $A$,

$$f_k(cA) = cf_k(A), \quad k = 1, 2, \cdots, n,$$

and that the functions $f_k$ are bounded on bounded sets; i.e., for all $n \times n$ matrices $A = (a_{ij})$ with $|a_{ij}| \leq c$ for all $1 \leq i, j \leq n$ with $i \neq j$, there exist positive constants $M_k(c)$ such that

$$f_k(A) \leq M_k(c), \quad k = 1, 2, \cdots, n.$$

The assumption of (1.7) is surely satisfied if each $f_k$ is continuous in its $n(n - 1)$ arguments $|a_{11}|, \cdots, |a_{n,n-1}|$. Note that assumptions of (1.6) and (1.7) are trivially satisfied for the $G$-generating family $\{f_1(A) = \sum_{j=1}^n a_{1j}, f_2(A) = \sum_{i=2}^n a_{i1}, \cdots, f_n(A) = \sum_{i=1}^n a_{ni}\}$ of (1.1).

In the subsequent material, for any subset $Q \subset \{1, 2, \cdots, n\}$, we use the notation $|Q|$ to denote the number of elements of $Q$.

2. Main result. Our main result is the following theorem.

**Theorem 1.** Let $D_1, D_2, \cdots, D_n$ be subsets of the set of all ordered pairs $(i, j)$, where $1 \leq i, j \leq n$ with $i \neq j$. Then, there exists a $G$-generating family $\{f_1, f_2, \cdots, f_n\}$ with each $f_k$ homogeneous (cf. (1.6)) and bounded on bounded sets (cf. (1.7)) satisfying

$$D_k = D(f_k), \quad k = 1, 2, \cdots, n,$$

if and only if for every subset $S \subset \{1, 2, \cdots, n\}$ with $|S| \geq 2$, for every cyclic permutation $\sigma$ of $S$, and for every nonempty subset $T \subset S$,

$$\left|\left\{1: i \in S \text{ and } (i, \sigma i) \in \bigcup_{k \in T} D_k\right\}\right| \geq |T|.$$  

**Proof.** Assuming (2.1), we first show the necessity of (2.2). Assume the contrary. Then, there exists an $S \subset \{1, 2, \cdots, n\}$, with $|S| \geq 2$, a cyclic permutation $\sigma$ of $S$, a nonempty subset $T \subset S$, and indices $i_1, \cdots, i_r \in S$ such that

$$\bigcup_{j=1}^r \{(i_j, \sigma i_j)\} \cup \{1, \sigma i\} \cap \bigcup_{k \in T} D(f_k),$$

with $r < t = |T|$. Next, given any $c$ with $0 < c < 1$, define the off-diagonal entries of the $n \times n$ matrix $A(c) = (a_{ij}(c))$ by

$$a_{ij}(c) = -c, \quad j = 1, 2, \cdots, r,$$

$$a_{i\sigma i} = -1, \quad i \in S, \quad i \neq i_1, i_2, \cdots, i_r,$$

$$a_{ik} = 0, \quad \text{otherwise } (k \neq l).$$

Note that any ordered pair $(i, \sigma i), i \in S, i \neq i_1, i_2, \cdots, i_r$, is, from (2.3), not in any $D(f_k), k \in T$. In other words, for any $k \in T, f_k(A(c))$ depends only on coefficients of
modulus \( \varepsilon \) or zero. Hence, by virtue of the assumption that each \( f_k \) is homogeneous, it follows (cf. (1.6)) that

\[
f_k(A(\varepsilon)) = c f_k(A(1)) \quad \text{for all } k \in T.
\]

Next, consider \( g(\varepsilon) \equiv \prod_{k \in S} f_k(A(\varepsilon)) \). This can also be written as

\[
g(\varepsilon) = \prod_{k \in T} f_k(A(\varepsilon)) \prod_{k \in S \setminus T} f_k(A(\varepsilon)) = \varepsilon' \prod_{k \in T} f_k(A(1)) \prod_{k \in S \setminus T} f_k(A(\varepsilon)),
\]

using the expression of (2.4). Because the off-diagonal entries of \( A(\varepsilon) \) are in modulus at most unity, it follows from the boundedness assumption on the \( f_k \) that (cf. (1.7))

\[
\prod_{k \in S \setminus T} M_k(1) \geq \prod_{k \in S \setminus T} f_k(A(\varepsilon)).
\]

Thus, as \( t > r \), we see that for all sufficiently small \( \varepsilon > 0 \),

\[
\varepsilon' > \varepsilon' \left( \prod_{k \in T} f_k(A(1)) \prod_{k \in S \setminus T} M_k(1) \right) \geq g(\varepsilon).
\]

Fixing \( \varepsilon \) sufficiently small, this means that we can find positive numbers \( c_k = c_k(\varepsilon) \), \( k \in S \), with

\[
c_k > f_k(A(\varepsilon)), \quad k \in S,
\]

and

\[
\varepsilon' = \prod_{k \in S} c_k > g(\varepsilon).
\]

Now, we define the diagonal entries of \( A(\varepsilon) \) as

\[
a_{j,j}(\varepsilon) = c_j, \quad j \in S,
\]

\[
a_{j,j}(\varepsilon) > f_j(A(\varepsilon)), \quad j \notin S.
\]

By means of our construction, we evidently have that \( a_{j,j} > f_j(A(\varepsilon)) \) for all \( j = 1, 2, \ldots, n \). But, as \( \{f_1, f_2, \ldots, f_n\} \) is by assumption a \( G \)-generating family, it follows from (1.3) that \( A(\varepsilon) \) is nonsingular. On the other hand, our construction gives us that \( A(\varepsilon) \) is (after a suitable permutation) the direct sum of an \( |S| \times |S| \) submatrix and a \( (n - |S|) \times (n - |S|) \) positive diagonal matrix. This \( |S| \times |S| \) submatrix, however, has determinant zero; the product of its diagonal entries is \( \varepsilon' \), and the only other nonzero term contributing to its determinant is \( \text{sgn} \sum a_{i,i} = -\varepsilon' \). Thus, \( A(\varepsilon) \) is singular, a contradiction which establishes the validity of (2.2).

Conversely, assume (2.2), and define the functions \( f_k \) by means of

\[
f_k(A) = \sqrt{n!} \sum_{(i,j) \in D_k} |a_{i,j}|, \quad k = 1, 2, \ldots, n.
\]

These functions \( f_k \) satisfy (2.1) and are obviously homogeneous and bounded on bounded sets.

It remains to show that \( \{f_1, f_2, \ldots, f_n\} \) is a \( G \)-generating family. From (1.3), it is sufficient to show that for any \( n \times n \) matrix \( A = (a_{i,j}) \) with \( |a_{i,j}| > f_k(A) \) for \( k = 1, 2, \ldots, n \), \( A \) is nonsingular. Thus, we assume that

\[
|a_{k,j}| > f_k(A), \quad k = 1, 2, \ldots, n.
\]
Consider any subset \( S \subset \{1, 2, \cdots, n\} \) with \(|S| \geq 2\) and any cyclic permutation \( \sigma \) on \( S \), and define the \(|S| \times |S|\) matrix \( B = (b_{i,k}) \), \( i, k \in S \), by
\[
b_{i,k} = \begin{cases} 
1 & \text{if } (i, \sigma i) \in D_k, \\
0 & \text{if } (i, \sigma i) \notin D_k.
\end{cases}
\]
Noting that \( B \) is a zero-one matrix, the assumption \( \phi \) of (2.2) is precisely the condition given by Philip Hall in his famous theorem on systems of distinct representatives (cf. Ryser [3]) that there exists a permutation \( \tau \) on \( S \) with \( b_{i,\tau i} = 1 \) for all \( i \in S \). Thus, by definition, \( (i, \sigma i) \in D_{\tau i} \) for all \( i \in S \); and from (2.5), we deduce that \( f_i(A) \geq \sqrt{n!} |a_{i,\sigma i}| \) for all \( i \in S \). Consequently, from (2.6),
\[
|a_{i,\sigma i}| > f_i(A) \geq \sqrt{n!} |a_{i,\sigma i}|,
\]
\( i \in S \).
Taking products in the above expression over all \( i \in S \), and noting that \(|S| \geq 2\), we have
\[
(2.7) \quad \prod_{i \in S} |a_{i,\sigma i}| > (n!)^{S/2} \prod_{i \in S} |a_{i,\sigma i}| \geq (n!) \prod_{i \in S} a_{i,\sigma i},
\]
for any subset \( S \subset \{1, 2, \cdots, n\} \) with \(|S| \geq 2\) and any cyclic permutation \( \sigma \) on \( S \). Recalling that the determinant of \( A \) is the sum of \( n! \) products of elements of \( A \), it is easy to see that (2.7) implies that \( A \) is nonsingular. This completes the proof.

We remark that with the assumption of (2.2), it can be shown by more intricate arguments that the functions \( f_i \) of (2.5), even with the factor \( \sqrt{n!} \) deleted, still form a \( G \)-generating family (cf. Hoffman [1]).

REFERENCES

