The Effect of Quadrature Errors in the Numerical Solution of Two-Dimensional Boundary Value Problems by Variational Techniques

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1. Introduction

In [8], quadrature schemes were prescribed for the numerical solution of a class of one-dimensional boundary value problems by variational technique described in [4]. This paper is the analogue of [8] for the real two-dimensional nonlinear boundary value problem

\[
\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = f(x, y, u), \quad (x, y) \in G
\]

\[u(x, y) = 0, \quad (x, y) \in \partial G,
\]

where \(G\) is a rectangle in the \((x, y)\)-plane. We point out that the results we present here extend easily to higher dimensions and to a larger class of nonlinear differential operators ([5, §3]), but, for ease of presentation, we will restrict ourselves to the problem (1.1). With certain assumptions on \(f(x, y, u)\), the unique solution of (1.1) can be approximated by applying the classical Ritz-Galerkin procedure to the variational formulation of the problem by minimizing over finite-dimensional subspaces ([5], [13, p. 188]). For sequences of piecewise-polynomial Hermite and spline subspaces, upper bounds for the rates of convergence of these approximations can be theoretically deduced from [1], [2], and [5]. These approximate solutions are, however, not precisely obtained in practical computation on a digital computer since certain integrals arising in the Ritz-Galerkin formulation are replaced by quadrature formulas.

The object of this paper is to investigate the errors introduced in the approximate solutions by such quadrature formulas. In particular, we shall obtain bounds for the errors introduced by such quadrature schemes, as they apply to finite-dimensional piecewise-polynomial Hermite and spline subspaces, and we shall determine when these quadrature errors are consistent with (i.e., the same order as) the approximation errors of the Ritz-Galerkin method. Numerical results based on such consistent quadrature schemes are also presented.

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2. Formulation of the Problem

Let us assume that \( G \) is the open rectangle \((a, b) \times (c, d)\) in the \((x, y)\)-plane; \( a < x < b, c < y < d \), and let \( \partial G \) denote its boundary. We define \( S \) to be the set of all continuous functions \( w(x, y) \) defined on \( G \), the closure of \( G \), such that \( w(x, y) \) is piecewise continuously differentiable over \( G \), and \( w(x, y) = 0 \) when \((x, y) \in \partial G\). The first eigenvalue, \( A_1 \), associated with the Helmholtz equation \( \Delta u + \lambda u = 0 \), over \( G \) is given by

\[
A_1 = \inf_{w \neq 0} \frac{\iint_{G} \left( \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) \, dx \, dy}{\iiint_{G} (w)^2 \, dx \, dy}.
\] (2.1)

The problem which we wish to consider is approximating the function \( \phi(x, y) \in C(G) \cap C^2(G) \), where \( \phi(x, y) \) satisfies (1.1). We assume that \( f(x, y, u) \in C^0(G \times R) \), and moreover that there exists a constant \( \gamma \) such that

\[
\frac{f(x, y, u_1) - f(x, y, u_2)}{u_1 - u_2} \geq \gamma > -A_1, \quad (x, y) \in G, \quad -\infty < u_1, u_2 < +\infty, \quad u_1 \neq u_2.
\] (2.2)

If \( \phi(x, y) \) is a solution of (1.1) when condition (2.2) holds, then it is shown in [3, p. 86] that \( \phi(x, y) \) (which is certainly in \( S \)) strictly minimizes the functional

\[
F[w] = \iint_{G} \left\{ \frac{1}{2} \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] + \iint_{G} f(x, y, \eta) \, d\eta \right\} \, dx \, dy
\] (2.3)

over \( S \), i.e.,

\[
F[\phi] = \inf_{w \in S} F[w].
\]

A consequence of this fact is that if a solution exists, it is unique. We will make the hypothesis that a solution of (1.1) exists; see [5] and [10] for further details.

As in the one-dimensional problem, we now consider a finite-dimensional subspace \( S_M \) of \( S \), of dimension \( M \), and minimize the functional (2.3) over \( S_M \). If the functions \( \{w_i(x, y)\}_{i=1}^{M} \) form a basis for \( S_M \), then any element in \( S_M \) can be written as

\[
w(x, y) = \sum_{j=1}^{M} u_j w_j(x, y).
\]

We know from [5, Theorem 3.3] that, given any finite-dimensional subspace \( S_M \) of
there exists in $S_M$, one and only one $\hat{\phi}_1(x, y)$ such that

$$F[\phi] = \inf_{w \in S_M} F[w].$$

In order to minimize the functional $F[w]$ over $S_M$ spanned by $(w_i(x, y))_{i=1}^M$, the following system of equations must be solved:

$$Au + k(u) = 0,$$

where $u = (u_1, u_2, \ldots, u_M)^T$, and where $A = (a_{i,j})$ is an $M \times M$ real, symmetric matrix, and $k(u) = (k_1(u), \ldots, k_M(u))^T$ is a column vector, which we now define: letting

$$\langle w, v \rangle = \iint_G \left\{ \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial x} \right) + \left( \frac{\partial w}{\partial y} \frac{\partial v}{\partial y} \right) \right\} \, dx \, dy, \quad w, v \in S,$$

then

$$a_{i,j} = \langle w_i, w_j \rangle, \quad 1 \leq i, j \leq M,$$

and

$$k_i(u) = \int_G f(x, y, \sum_{j=1}^M u_j w_j) w_i \, dx \, dy, \quad 1 \leq i \leq M.$$

Denoting the solution of (2.4) by $\hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_M)^T$, the approximation to the solution $\phi(x, y)$ of (1.1) is then

$$\hat{\phi}_1(x, y) = \sum_{j=1}^M \hat{u}_j w_j(x, y).$$

The inner product (2.5) defined on set $S$ induces the norm

$$\|w\|_D = \langle w, w \rangle^{1/2} = \left( \iint_G \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) \, dx \, dy \right)^{1/2}$$

on $S$. This norm is easily seen to be equivalent to the Sobolev norm [15, p. 55]

$$\|w\|_{1, 2} = \left\{ \iint_G \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \, dx \, dy \right\}^{1/2}.$$

The quantity

$$\|w\|_\gamma = \left\{ \iint_G \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 + \gamma (w)^2 \right] \, dx \, dy \right\}^{1/2},$$

for $\gamma > -\lambda_1$, is also a norm on $S$ ([3, p. 97]) and is associated with the inner product

$$\langle w, v \rangle_\gamma = \iint_G \left\{ \frac{\partial w}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial v}{\partial y} + \gamma w v \right\} \, dx \, dy, \ w, v \in S.$$
This norm is equivalent to \( \| \cdot \|_D \) in (2.8) and is the norm basically used in the sections to follow. Note that \( \| \cdot \|_D = \| \cdot \|_0 \).

Let \( \{ S_{M_i} \}_{i=1}^\infty \) be a sequence of finite-dimensional subspaces of \( S \) and denote by \( \hat{w}_{M_i} \), the unique element which minimizes \( F[w] \) over \( S_{M_i} \). From [5, Corollary 3.1] we know that if \( \lim_{i \to \infty} \{ \inf_{w \in S_{M_i}} \| w - g \|_0 \} = 0 \) for all \( g \in S \), \( \partial f(x, y, u)/\partial u \in C^0(G \times R) \) and if there exist two constants \( \gamma \) and \( K \) such that

\[
-A_1 < \gamma \leq \frac{\partial f(x, y, u)}{\partial u} \leq K, \quad (x, y) \in \bar{G}, \quad -\infty < u < +\infty,
\]

then

\[
\lim_{i \to \infty} \| \phi - \hat{w}_{M_i} \|_0 = 0.
\]

The hypothesis that \( \partial f(x, y, u)/\partial u \) be bounded is rather restrictive. We can make other requirements on the function \( f \) which enable us to omit the hypothesis that \( \partial f/\partial u \) be bounded above and to obtain the convergence of the sequence \( \{ \hat{w}_{M_i} \}_{i=1}^\infty \). In [5, Theorems 4.1 and 4.2] it is shown that one can derive an a priori pointwise bound for the solution \( \phi(x, y) \) of (1.1) when one of the following conditions hold:

\[
\lim_{|u| \to \infty} \inf_{(x, y) \in \bar{G}} \frac{f(x, y, u)}{u} \geq 0, \quad (x, y) \in \bar{G},
\]

\[
\frac{\partial f(x, y, u)}{\partial u} \geq \gamma > -\frac{1}{q}, \quad (x, y) \in \bar{G}, \quad -\infty < u < +\infty,
\]

where \( q \) is a positive quantity determined as follows. Let \( \Psi(x, y) \) be the (unique) solution of \( \Delta u(x, y) = -1, (x, y) \in \bar{G}; u(x, y) = 0, (x, y) \in \partial G \); then \( q = \sup_{(x, y) \in \bar{G}} |\Psi(x, y)| > 0 \).

Consider the problem (1.1) when either condition (2.10) or (2.11) holds. Assume that \( M \) is the constant such that the unique solution \( \phi(x, y) \) of (1.1) satisfies \( |\phi(x, y)| \leq M \) for all \( (x, y) \in \bar{G} \). Then, as noted in [3, p. 96], solving (1.1) is equivalent to solving

\[
\left\{ \begin{aligned}
\Delta u(x, y) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y, u), \quad (x, y) \in \bar{G}, \\
\quad u(x, y) &= 0, \quad (x, y) \in \partial G,
\end{aligned} \right.
\]

where

\[
g(x, y, u) = \begin{cases} 
    f(x, y, M), & M < u \\
    f(x, y, u), & |u| \leq M, \\
    f(x, y, -M), & u < -M.
\end{cases}
\]

Therefore, suppose \( \partial f(x, y, u)/\partial u \in C^0(G \times R) \). If \( f(x, y, u) \) does not satisfy the condition that \( \partial f/\partial u \) is bounded above but it does satisfy either (2.10) or (2.11), then we can solve the corresponding problem (2.12) which satisfies (2.9) and the sequence
\{w_M, (x, y)\}_{i=1}^\infty \text{ converges in the norm } \| \cdot \|_D \text{ to } \phi(x, y), \text{ the unique solution of (1.1), as well as (2.12). Hence, we will assume that } \partial f/\partial u(x, y, u) \in C^0(\bar{G} \times R), \text{ and then either } \partial f/\partial u \text{ is bounded above or either (2.10) or (2.11) holds.}

For any subspace \(S_M\) of \(S\), it can be shown (cf. [3, p. 100], [5, Theorem 4.4]) that there exists a positive constant \(C\), independent of the choice of \(S_M\), such that the following error bound is valid:

\[ \| \hat{\psi} - \phi \|_D \leq C \inf_{w \in S_M} \| w - \phi \|_D, \tag{2.13} \]

where \(\hat{\psi}\) is such that \(F[\hat{\psi}] = \inf_{w \in S_M} F[w]\). This implies that rigorous bounds can be deduced from the two-dimensional interpolation error estimates of [1] and [2] for the quantities \(\| \hat{\psi} - \phi \|_D\) and \(\| \hat{\psi} - \phi \|_\gamma\). Unfortunately, Sobolev’s Imbedding Theorem [15, p.174] in two dimensions for this case does not allow us to deduce error bounds in the uniform norm, as was the case in one dimension.

Let us consider the system of equations (2.4) which must be solved. For piecewise-polynomial Hermite and spline subspaces, the matrix entries \(a_{i,j}\) of the matrix \(A\) are easily computed since this involves only the integration of polynomials, which can be easily automated on a digital computer. The quantities \(k(u)\) in (2.7) are more difficult to compute because \(f\) is not in general a piecewise-polynomial. This prompts us to use a quadrature scheme to evaluate the quantities in (2.7), which then generates a new system of nonlinear equations

\[ 0 = Au + \tilde{k}(u), \tag{2.14} \]

where \(\tilde{k}(u)\) is obtained by applying quadrature scheme to \(k(u)\). The solution \(\tilde{u}\) of (2.14) in turn generates a new function

\[ \hat{\psi} = \sum_{i=1}^M \tilde{u}_i w_i \text{ in } S_M. \]

In the next two sections, we shall discuss the choice of quadrature schemes for a given sequence of piecewise-polynomial subspaces \(\{S_M\}_{i=1}^\infty\) of \(S\), so that the approximations \(\{\hat{\psi}_M\}_{i=1}^\infty\), determined successively from (2.14), have the same general order of accuracy as the theoretical approximations \(\{\hat{\psi}_M\}_{i=1}^\infty\), determined successively from (2.4).

Without loss of generality, from here on in our discussion we will assume that \(G\) is the open unit square \((0, 1) \times (0, 1)\).

3. Linear Case

Let us suppose that the function \(f\) in (1.1) is not a function of \(u\). In this case, \(f(x, y)\) satisfies (2.2) with \(\alpha = 0\), and \(\partial f/\partial u\) is bounded above by zero. The integrals of
(2.7) are also independent of \( u \), and in this case, we have
\[
k_i = \int \int_G f(x, y) w_i(x, y) \, dx \, dy = L \left[ f(x, y) w_i(x, y) \right], \quad 1 \leq i \leq M, \quad (3.1)
\]
where the integral in (3.1) is regarded as a bounded linear functional, \( L \), on \( C^1(G) \). With the subspace \( S_M \), we associate a linear functional \( L_M \) which is to approximate \( L \), and we define
\[
\hat{k}_i = L_M \left[ f(x, \hat{y}) w_i(x, \hat{y}) \right], \quad 1 \leq i \leq M \quad (3.2)
\]
as the approximations of \( k_i \) in (3.1). The matrix problem of (2.4) now reduces to
\[
A u + k = 0 \quad (3.3)
\]
and the use of the approximate linear functional \( L_M \) gives the associated matrix problem
\[
A u + \hat{k} = 0. \quad (3.4)
\]
As previously noted, \( A \) is a real symmetric matrix, and since we can easily verify that
\[
u^T A u = \left\| \sum_{j=1}^M u_j w_j \right\|_D^2, \quad (3.5)
\]
\( A \) is then also positive definite. Hence, each of the matrix problems (3.3) and (3.4) admits a unique solution, denoted by \( \hat{u} \) and \( \hat{u} \) respectively, and the associated functions in \( S_M \) are \( \hat{w}_M = \sum_{j=1}^M \hat{u}_j w_j \) and \( \hat{w}_M = \sum_{j=1}^M \hat{u}_j w_j \). From (3.3) and (3.4), it follows that
\[
A (\hat{u} - \bar{u}) = (\hat{k} - \bar{k}), \text{ and premultiplying by } (\hat{u} - \bar{u})^T \text{ and using (3.5) gives}
\]
\[
(\hat{u} - \bar{u})^T A (\hat{u} - \bar{u}) = \| \hat{w}_M - \bar{w}_M \|_D^2 = (\hat{u} - \bar{u})^T (\hat{k} - \bar{k}).
\]
Using the definitions of the functionals \( L \) and \( L_M \), the last quantity above can be expressed as
\[
(L_M - L) \left[ f(x, y) (\hat{w}_M(x, y) - \hat{w}_M(x, y)) \right],
\]
and thus
\[
\| \hat{w}_M - \bar{w}_M \|_D^2 = (L_M - L) \left[ f(x, y) (\hat{w}_M(x, y) - \hat{w}_M(x, y)) \right]. \quad (3.6)
\]
This equation will be used repeatedly in this section.

Our object now is to bound \( \| \hat{w}_M - \bar{w}_M \|_D \) for certain quadrature schemes \( L_M \), after making further assumptions on \( f \) and the subspace \( S_M \). Let
\[
\Delta_x : 0 = x_0 < x_1 < \cdots < x_{N_x + 1} = 1, \quad \Delta_y : 0 = y_0 < y_1 < \cdots < y_{N_y + 1} = 1
\]
be a partition of the unit square \( G \) and denote by \( \Delta \) the set of points \((x_i, y_j)\), \( 0 \leq i \leq N_x + 1, 0 \leq j \leq N_y + 1 \). We will always assume that \( \Delta \leq \delta_0 \Delta \), where \( \delta_0 \) is a fixed positive number, and where
\[
\bar{\Delta} = \max_{0 \leq i \leq N_x, 0 \leq j \leq N_y} \{ |x_{i+1} - x_i|, |y_{j+1} - y_j| \}
\]
and $\mathcal{A}$ is the corresponding minimum. We now restrict our attention to subspaces $S_M(\mathcal{A})$ of $S$ of piecewise-polynomial functions. More precisely, for any $w(x, y) \in S_M(\mathcal{A})$, $w(x, y)$ is a piecewise-polynomial of degree $n_0$ in each variable and a polynomial in each cell
\[
[x_i, x_{i+1}] \times [y_j, y_{j+1}], \; \; 0 \leq i \leq N_x, \; \; 0 \leq j \leq N_y, \; \; \text{of} \; \; \mathcal{A}.
\]
Such subspaces include the Hermite and spline subspaces as special cases (cf. [1] and [2]). We also assume that the function $f$ of (1.1) is such that $f(x, y) \in C^{m_0}(\mathcal{G}) \times \mathcal{G}$, $0 \leq i \leq N_x, \; 0 \leq j \leq N_y$. This latter hypothesis is of course valid if $f(x, y) \in C^{m_0}(\mathcal{G})$, but it also holds for functions $f(x, y)$ whose $m_0$-th derivative is piecewise continuous on $G$, with points of discontinuity on the boundaries of the cells of $\mathcal{A}$. The important point is that since $f$ is given, the quantity $m_0$ and the possible points of discontinuity of the partial derivatives of $f$ can be determined directly.

As our first choice for the bounded linear functional $L_M$, consider a quadrature scheme of the form
\[
\int_{x_0}^{x_m} \int_{y_0}^{y_m} \sigma(x, y) \, dy \, dx = \sum_{i, j = 0}^{m} \alpha_{ij} \sigma(\tau_i, \eta_j), \tag{3.7}
\]
where $x_0 < \tau_0 < \tau_1 < \cdots < \tau_m \leq x_m$, $y_0 < \eta_0 < \eta_1 < \cdots < \eta_m \leq y_m$ are selected points of $[x_0, x_m]$ and $[y_0, y_m]$. Given any $\sigma(x, y) \in C^{m_0}(\mathcal{X}_x, x_m) \times [y_0, y_m]$ and $m_0$ determined from $f$, it is always possible to select a quadrature scheme of the form (3.7) such that the quadrature error of (3.7) satisfies
\[
\left| \sum_{i, j = 0}^{m} \alpha_{ij} \sigma(\tau_i, \eta_j) - \int_{x_0}^{x_m} \int_{y_0}^{y_m} \sigma(x, y) \, dy \, dx \right| \leq K \tau^{m_0 + 2} \max_{i = 0, m_0} \left\{ \left\| \frac{\partial^{m_0} \sigma}{\partial x^i \partial y^{m_0-i}} \right\|_{L^\infty(\mathcal{X}_x, x_m) \times [y_0, y_m]} \right\}, \tag{3.8}
\]
where $\tau \equiv \max(x_m - x_0, y_m - y_0)$, and where $K$ is independent of $(y_m - y_0)$ and $(x_m - x_0)$. Quadrature schemes such as (3.7)–(3.8) are very easily generated by taking in essence the Cartesian product of one-dimensional quadrature schemes of the form
\[
\int_{y_0}^{y_m} \sigma(t) \, dt = \sum_{i = 0}^{m} \alpha_i \sigma(\tau_i),
\]
with
\[
\left| \sum_{i = 0}^{m} \alpha_i \sigma(\tau_i) - \int_{y_0}^{y_m} \sigma(t) \, dt \right| \leq K (y_m - y_0)^{m_0+1} \left\| \frac{d^{m_0} \sigma}{d x^{m_0}} \right\|_{L^\infty(y_0, y_m)},
\]
and applying it twice to
\[
\int_{x_0}^{x_m} \int_{y_0}^{y_m} \sigma(x, y) \, dy \, dx.
\]
See also [11, ch. 4]. Writing $k_i$ in (3.1) as

$$
k_i = \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} \int_{x_{l+1}'} \int_{y_{k+1}'} f(x, y) w_i(x, y) \, dy \, dx,
$$

we apply (3.7) on each cell $[x_{l}, x'_{l+1}] \times [y_{k}, y'_{k+1}]$, and this in turn defines the linear functional $L_M$ of (3.2) which we will denote as

$$
L_M[f(x, y) w_i(x, y)] = \tilde{k}_i = \sum_{l=0}^{M_x} \sum_{k=0}^{M_y} \beta_{l,k} f(x_l, y_k) w_i(x_l, y_k)
$$

(3.9)

for simplicity of notation. This brings us to

**THEOREM 1.** Assuming that $f$ in (1.1) is independent of $u$, let $\partial^{m_0} f(x, y)/(\partial x)^{m_0}$ and $\partial^{m_0} f(x, y)/(\partial y)^{m_0}$ be continuous on each cell of $\Delta$. If $Q$ is any collection of quasi-uniform partitions of $\bar{\Delta} = [0, 1] \times [0, 1]$, i.e., there exists a fixed positive constant $\delta_0$ such that $\delta_0 \Delta \supset \Delta$ for all $\Delta \in Q$, and if for each $\Delta \in Q$, $S_M(\Delta)$ is any finite-dimensional subspace of $S$ such that for any $v(x, y) \in S_M(\Delta)$, $v(x, y)$ is a polynomial of degree $n_0$ in each variable on each cell of $\Delta$, then for $m_0 \geq n_0 + 2$, the linear functional $L_M$ defined in (3.9) is such that

$$
\|\hat{w}_M - \hat{w}_M\|_D \leq B(\bar{\Delta})^{m_0-m_0-2},
$$

(3.10)

where $B$ is a constant independent of $\Delta$.

**Proof.** Expressing $(L_M - L)[f(\hat{w}_M - \hat{w}_M)]$ in (3.6) as a sum of terms and applying (3.8) to each of these terms gives

$$
\|\hat{w}_M - \hat{w}_M\|_D^2 = (L_M - L)[f(\hat{w}_M - \hat{w}_M)] \\
\leq \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} K(\bar{\Delta})^{m_0+2} \max_{i=0, m_0} \left| \frac{\partial^{m_0} f(x_l, y_k)}{(\partial x)^{m_0}} \right|_{L^1}, \tag{3.11}
$$

where

$$
\|\cdot\|_{L^1} \equiv \|\cdot\|_{L^1([x_{l+1}', y_{k+1}'])}
$$

and $K$ is independent of $\Delta$. By hypothesis, there exists a constant $C_1$, independent of $\Delta$, such that

$$
\max_{0 \leq j \leq N_x} \max_{0 \leq i \leq N_y} \left\{ \frac{\partial^{j_1} f(x, y)}{(\partial x)^{j_1}} \right\}_{L^1} \leq C_1,
$$

for $0 \leq j_1, j_2 \leq m_0$. Consequently, using the Leibnitz formula for differentiation of a product, the sum of (3.11) is bounded above by

$$
C_1 K(\bar{\Delta})^{m_0+2} \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} \max_{i=0} \left\{ \sum_{j=0}^{m_0-1} \left( \sum_{s=0}^{i} \binom{i}{s} (m_0 - i) \right) \left\| \frac{\partial^{r+s} f(x_l, y_k)}{(\partial x)^r (\partial y)^s} \right\|_{L^1} \right\}.
$$

(3.12)
By the assumption that the elements of \( S_M(A) \) are piecewise-polynomials of degree \( n_0 \) in each variable, the upper limits on the sums on \( r \) and \( s \) in (3.12) are at most \( n_0 \). Since \( \hat{w}_M - \hat{w}_M \) is a polynomial in each cell of \( A \) then, by a theorem of Markov [12, p. 138], there exists a constant \( C_2 \), independent of \( A \), such that

\[
\left| \frac{\partial^{i+j}(\hat{w}_M - \hat{w}_M)}{\partial x^i \partial y^j} \right|_{L^\infty_{i,j,k}} \leq C_2 \frac{\|\hat{w}_M - \hat{w}_M\|_{L^\infty_{i,j,k}}}{(\hat{A})^{i+j}}.
\]  

(3.13)

for all \( 0 \leq i \leq n_0, 0 \leq l \leq N_x \), and \( 0 \leq k \leq N_y \). Using (3.13), we can deduce from (3.11) and (3.12) that there is a constant \( C_3 \), independent of \( A \), such that

\[
\|\hat{w}_M - \hat{w}_M\|_{L^2} \leq C_3 (N_x + 1) (N_y + 1) (\hat{A})^{n_0+2-n_0} \|\hat{w}_M - \hat{w}_M\|_{L^\infty(G)}.
\]  

(3.14)

Realizing that \( \max(N_x+1, N_y+1) \leq \hat{A}^{-1} \) and since we are assuming \( \hat{A} \leq \delta_0 \hat{A} \), we are assured from (3.14) that there exists a constant \( C_4 \) such that

\[
\|\hat{w}_M - \hat{w}_M\|^2_{L^2} \leq C_4 (\hat{A})^{n_0-n_0} \|\hat{w}_M - \hat{w}_M\|_{L^\infty(G)}.
\]  

(3.15)

We now wish to relate \( \|\hat{u} - \hat{w}\|_{L^\infty(G)} \) and \( \|\hat{w} - \hat{w}_M\|_{L^\infty(G)} \), using the fact that \( \hat{w}_M - \hat{w}_M \)

\[
\int_{-1}^{+1} \frac{dp_n(x)}{dx} \, dx \leq An^2 \int_{-1}^{+1} |p_n(x)| \, dx
\]

and hence by a change of variable

\[
\int_{a}^{b} \frac{dq_n(x)}{dx} \, dx \leq 2An^2 \int_{a}^{b} |q_n(x)| \, dx.
\]

for any polynomial \( q_n(x) \) of degree \( n \) defined on \([a, b]\). Therefore, for \( z \in [a, b] \), we have the string of inequalities

\[
|q_n(z)| - |q_n(a)| \leq |q_n(z) - q_n(a)| = \left| \int_{a}^{z} \frac{dq_n(x)}{dx} \, dx \right| \leq \left| \int_{a}^{b} \frac{dq_n(x)}{dx} \, dx \right| \leq 2An^2 \int_{a}^{b} |q_n(x)| \, dx.
\]

(3.16)
Suppose we have a piecewise-polynomial \( \sigma(x) \) defined on \([0,1]\), where \( \Delta_0:0<x_0<x_1<\cdots<x_{N+1}=1 \) is a partition of \([0,1]\), with \( \Delta_0 \leq \delta_1 \Delta_0, \delta_1 > 0 \), and \( \sigma(x) \) is a polynomial of degree \( n \) on \([x_i, x_{i+1}]\), \( 0 \leq i \leq N \). Also, we assume \( \sigma(0)=0 \) and denote by \( \sigma_i(x) \) the function \( \sigma(x) \) restricted to \([x_i, x_{i+1}]\). Using (3.16) on \( \sigma_0(x) \) with \( a=x_0=0 \) and \( b=x_1 \), we see that

\[
\|\sigma_0(x)\|_{L^\infty[x_0, x_1]} \leq \frac{2An^2}{(x_1-x_0)} \int_{x_0}^{x_1} |\sigma_0(x)| \, dx.
\]

From this inequality, we see that

\[
|\sigma_0(x_1)| \leq \frac{2An^2}{(x_1-x_0)} \int_{x_0}^{x_1} |\sigma_0(x)| \, dx,
\]

so using (3.16) on \( \sigma_1(x) \) with \( a=x_1 \) and \( b=x_2 \) we obtain

\[
\|\sigma_1(x)\|_{L^\infty[x_1, x_2]} \leq \frac{2An^2}{(x_1-x_0)} \int_{x_0}^{x_1} |\sigma_0(x)| \, dx + \frac{2An^2}{(x_2-x_1)} \int_{x_1}^{x_2} |\sigma_1(x)| \, dx.
\]

Continuing, we see that

\[
\|\sigma(x)\|_{L^\infty[0, 1]} \leq \sum_{j=0}^{N} \frac{2An^2}{(x_{j+1}-x_j)} \int_{x_j}^{x_{j+1}} |\sigma_j(x)| \, dx
\]

\[
\leq \frac{2An^2}{\Delta_0} \int_{0}^{1} |\sigma(x)| \, dx.
\]

Hence, using the Cauchy-Schwarz inequality,

\[
\|\sigma(x)\|_{L^2(0, 1)} \leq \frac{2An^2}{\Delta_0} \|\sigma(x)\|_{L^2(0, 1)}.
\]

Since \( \Delta_0 \leq \delta_1 \Delta_0 \), we can let \( \beta_0 = 2An^2 \delta_1 \) and we have

\[
\|\sigma(x)\|_{L^2(0, 1)} \leq \frac{\beta_0}{\Delta_0} \|\sigma(x)\|_{L^2(0, 1)}.
\]
Applying this to \( \hat{w} (x, y) - \hat{w} (x, y) \), we see that
\[
\| \hat{w} - \hat{w} \|_{L^{\infty} (\mathcal{G})} = \sup_{0 \leq x \leq 1} \sup_{0 \leq y \leq 1} | \hat{w} (x, y) - \hat{w} (x, y) | \\
\leq \sup_{0 \leq x \leq 1} \beta \frac{\beta}{\mathcal{A}} \sqrt{\int_{0}^{1} \int_{0}^{1} | \hat{w} - \hat{w} |^2 \, dy} \\
\leq \frac{\beta \beta}{\mathcal{A}} \sqrt{\int_{0}^{1} \int_{0}^{1} \left( \sup_{0 \leq x \leq 1} | \hat{w} - \hat{w} | \right)^2 \, dy} \\
\leq \frac{\beta \beta}{\mathcal{A}} \sqrt{\int_{0}^{1} \left( \frac{\beta}{\mathcal{A}} \sqrt{\int_{0}^{1} | \hat{w} - \hat{w} |^2 \, dx} \right)^2 \, dy},
\]
so that
\[
\| \hat{w} - \hat{w} \|_{L^{\infty} (\mathcal{G})} \leq \frac{\beta \beta}{\mathcal{A}} \| \hat{w} - \hat{w} \|_{L^2 (\mathcal{G})}.
\]
Hence, (3.15) becomes
\[
\| \hat{w} - \hat{w} \|_D^2 \leq C_4 (\mathcal{A})^{\mu_0 - \mu_0 - 2} \frac{\beta \beta}{\mathcal{A}} \| \hat{w} - \hat{w} \|_{L^2 (\mathcal{G})},
\]
and since \( \| \cdot \|_{L^2 (\mathcal{G})} \leq \| \cdot \|_{L^1, 2} \leq C_0 \| \cdot \|_D \) for some \( C_0 \), let \( B = C_4 \beta \beta \mathcal{C} \) and we have
\[
\| \hat{w} - \hat{w} \|_D \leq B (\mathcal{A})^{\mu_0 - \mu_0 - 2},
\]
where \( B \) is independent of \( \mathcal{A} \).

If we have a sequence \( \{ S_{M_i} (\mathcal{A}_i) \}_{i=1}^{\infty} \) of finite-dimensional subspaces of \( S \) with \( \mathcal{A}_i \leq \mathcal{A}_1 \mathcal{A}_i \) such that the elements of any \( S_{M_i} (\mathcal{A}_i) \) are piecewise-polynomials of fixed degree \( n_0 \), and if \( \lim_{i \to \infty} \mathcal{A}_i = 0 \), then from (3.10) if \( m_0 \), dependent only on \( f \), satisfies \( m_0 > n_0 + 2 \), we evidently have
\[
\lim_{i \to \infty} \| \hat{w}_{M_i} - \hat{w}_{M_i} \|_D = 0.
\]
Hence, the quadrature error, introduced by computing \( \hat{w}_{M_i} \) rather than \( \hat{w}_{M_i} \), tends to zero with \( i \). This error, however, may or may not be small, relative to \( \| \hat{w}_{M_i} - \phi \|_D \).

This brings us to

**DEFINITION 1.** Let \( Q \) be a collection of partitions \( \mathcal{A} \) of \( \mathcal{G} \), and for each \( \mathcal{A} \in Q \), let \( S_{M} (\mathcal{A}) \) be a finite-dimensional subspace of \( S \) consisting of elements which are polynomials of fixed degree \( n_0 \) in each cell defined by \( \mathcal{A} \), and let \( \hat{w}_M \), the function which minimizes \( F[\hat{w}] \) of (2.3) over \( S_{M}(x) \), satisfy
\[
\| \hat{w}_M - \phi \|_W \leq K (\mathcal{A})^{\gamma} \quad \text{for all} \quad \mathcal{A} \in Q,
\]
(3.17)
where \( K \) and \( l \) are positive constants independent of \( \Delta \), \( \varphi(x, y) \) is the solution of (1.1), and \( \| \cdot \|_x \) is some norm on the space \( S \). Then, the choice of linear functionals in (3.2) is consistent in the norm \( \| \cdot \|_N \) with the bounds of (3.17) if there exists a constant \( K_0 \), independent of \( \Delta \), such that

\[
\| \hat{w}_M - \tilde{w}_M \|_N \leq K_0 (\bar{\Delta})^l \quad \text{for all} \quad \Delta \in Q.
\]

From the triangle inequality, the bounds of (3.17) for the norm \( \| \cdot \|_D \), and the results of Theorem 1, it follows that

\[
\| \hat{w} - \varphi \|_D \leq \| \hat{w} - \tilde{w} \|_D + \| \tilde{w} - \varphi \|_D \leq B (\bar{\Delta})^{m_0 - n_0 - 2} + K (\bar{\Delta})^l, \quad \Delta \in Q.
\]

Hence, \( m_0 - n_0 - 2 \geq l \) gives a consistent choice of functionals in (3.9) in the norm \( \| \cdot \|_D \) which preserves the asymptotic accuracy of (3.17) in this norm. Note that even if this choice is not consistent in the norm \( \| \cdot \|_D \), i.e., if \( l < m_0 - n_0 - 2 \), it follows that when the collection \( Q \) is a sequence of partitions \( \{ \Delta_i \}_{i=1}^\infty \) with \( \bar{\Delta}_i \leq \delta_i \Delta_i \) for some fixed positive \( \delta_i \), then the associated sequence \( \{ \tilde{w}_M \}_{i=1}^\infty \) converges in the norm \( \| \cdot \|_D \) to \( \varphi(x, y) \) as \( i \to \infty \) when \( \lim_{i \to \infty} \bar{\Delta}_i = 0 \).

Another way to approximate the quantities \( k_i \), \( 1 \leq i \leq M \), in (3.1) is by replacing \( f(x, y) \) by an interpolate, calling it \( \tilde{f}(x, y) \), and evaluating \( \int_{\Gamma} \tilde{f}(x, y) w_i(x, y) \, dx \, dy \) exactly. We assume that we have a partition \( \Delta \) determined by the partitions \( \Delta_x: 0 = x_0 < x_1 < \cdots < x_{N_x+1} = 1 \) and \( \Delta_y: 0 = y_0 < y_1 < \cdots < y_{N_y+1} = 1 \) of the \( x \) and \( y \) axis. Note that if \( \{w_i(x, y)\}_{i=1}^M \) and \( \tilde{f}(x, y) \) are polynomials on each cell \( [x_i, x_{i+1}] \times [y_j, y_{j+1}] \) of \( \Delta \), then the integral \( \int_{\Gamma} \tilde{f}(x, y) w_i(x, y) \, dx \, dy \) is simply the sum of integrals of polynomials over the cells of \( \Delta \), and is therefore easy to calculate on a digital computer. We now determine how accurate an approximation \( \tilde{f}(x, y) \) must be to \( f(x, y) \) in order for this type of quadrature scheme to be useful in our variational technique.

The interpolation of \( f(x, y) \) in (3.1) by \( \tilde{f}(x, y) \) would generate a system of equations (3.4) where

\[
\tilde{k}_i = L_M [f w_i] = L (\tilde{f} w_i) = \int_{\Gamma} \tilde{f}(x, y) w_i(x, y) \, dx \, dy, \quad 1 \leq i \leq M. \tag{3.18}
\]

This in turn again serves to define \( \tilde{w}_M(x, y) = \sum_{j=1}^M \tilde{u}_j w_j(x, y) \) from the solution of (3.4).

**THEOREM 2.** Assuming that \( f \) in (1.1) is independent of \( u \), let \( \Delta \) be any partition of \( G = [0,1] \times [0,1] \) such that \( f \) is continuous on each cell of \( \Delta \) and let \( S_M(\Delta) \) be any finite-dimensional subspace of \( S \) such that for every \( v(x, y) \in S_M(\Delta) \), \( v(x, y) \) is a polynomial of degree \( n_0 \) in each variable on each cell of \( \Delta \). If \( \tilde{f}_M(x, y; \Delta) \) is a continuous piecewise-polynomial interpolate of \( f(x, y) \) such that \( \tilde{f}(x, y; \Delta) \) is a polynomial on each cell defined by \( \Delta \), let \( L_M \) be the associated linear functional of (3.18). Then, there
exists a constant $K$, independent of $A$, such that
\begin{equation}
\| \tilde{w}_M - \hat{w}_M \|_D \leq K \| \tilde{f}_M - f \|_{L^2(G)}.
\end{equation}

Proof. From (3.6), we have
\begin{equation}
\| \tilde{w}_M - \hat{w}_M \|_D^2 = \int_{G} \left[ \int_{G} \left[ \tilde{f}_M(x, y; A) - f(x, y) \right] \left[ \tilde{w}_M(x, y) - \hat{w}_M(x, y) \right] dx \right] dy,
\end{equation}
and using the Cauchy-Schwarz inequality gives
\begin{equation}
\| \tilde{w}_M - \hat{w}_M \|_D^2 \leq \| \tilde{f}_M - f \|_{L^2(G)} \| \tilde{w} - \hat{w} \|_{L^2(G)}.
\end{equation}
Now $\| \tilde{w}_M - \hat{w}_M \|_{L^2(G)} \leq \| \tilde{w} - \hat{w} \|_{1, 2}$, and $\| \cdot \|_D$ is equivalent to $\| \cdot \|_{1, 2}$, so there exists a constant $K$ such that
\begin{equation}
\| \tilde{w}_M - \hat{w}_M \|_{L^2(G)} \leq K \| \tilde{w}_M - \hat{w}_M \|_D.
\end{equation}
Hence from (3.20),
\begin{equation}
\| \tilde{w}_M - \hat{w}_M \|_D \leq K \| \tilde{f}_M - f \|_{L^2(G)}.
\end{equation}
Q.E.D.

From inequality (3.19), it is now clear how the piecewise-polynomial interpolate is to be chosen so as to have a consistent quadrature scheme in some norm. For example, if $f(x, y) \in C^4(G)$, then the piecewise cubic Hermite interpolate $\tilde{f}(x, y)$, relative to a partition $A$ on $G$, satisfies (cf. [1])
\begin{equation}
\| f - \tilde{f} \|_{L^2(G)} \leq K_1 (A)^4.
\end{equation}

4. Nonlinear Case

We now define what we mean by a consistent quadrature scheme for the general problem (1.1), where $f$ is a function of $u$, as well as $x$ and $y$.

If we approximate the integrals $k_i(u)$, $i \leq M$, in (2.7) by a quadrature scheme and denote these approximations by $\tilde{k}_i(u)$, we have the following new system to solve:
\begin{equation}
Au + \tilde{k}(u) = 0.
\end{equation}

Naturally, since $f$ is dependent on $u$, we are not assured that the system (4.1) has a unique solution. As in the previous section, we will denote by $\tilde{w}_M(x, y)$ the approximation generated by the system (2.4) using subspace $S_M$.

Let $A_x: 0 = x_0 < x_1 < \cdots < x_{N_x+1} = 1$, $A_y: 0 = y_0 < y_1 < \cdots < y_{N_y+1} = 1$ be a partition of the unit square $G$ and, as in §3, denote by $A$ the set of points $(x_i, y_j)$, $0 \leq i \leq N_x + 1$, $0 \leq j \leq N_y + 1$. Again assume $A \leq \delta_0 A$, where $\delta_0$ is a fixed positive constant, and we restrict our attention to subspaces $S_M(A)$ of $S$ of piecewise-polynomial functions. Writing $k_i(u)$ in (2.7) as the sum
\begin{equation}
k_i(u) = \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} \sum_{j=1}^{M} \int_{x_l}^{x_{l+1}} \int_{y_k}^{y_{k+1}} f(x, y, \sum_{j=1}^{M} u_j w_j) w_l dy dx,
\end{equation}
and using the Cauchy-Schwarz inequality gives
\begin{equation}
\| k_i - \tilde{k}_i \|_D \leq K \| k_i - \tilde{k}_i \|_{L^2(G)}.
\end{equation}
Now $\| k_i - \tilde{k}_i \|_{L^2(G)} \leq \| k_i - \tilde{k}_i \|_{1, 2}$, and $\| \cdot \|_D$ is equivalent to $\| \cdot \|_{1, 2}$, so there exists a constant $K$ such that
\begin{equation}
\| k_i - \tilde{k}_i \|_{L^2(G)} \leq K \| k_i - \tilde{k}_i \|_D.
\end{equation}
Hence from (3.20),
\begin{equation}
\| k_i - \tilde{k}_i \|_D \leq K \| f - \tilde{f} \|_{L^2(G)}.
\end{equation}
Q.E.D.

From inequality (3.19), it is now clear how the piecewise-polynomial interpolate is to be chosen so as to have a consistent quadrature scheme in some norm. For example, if $f(x, y) \in C^4(G)$, then the piecewise cubic Hermite interpolate $\tilde{f}(x, y)$, relative to a partition $A$ on $G$, satisfies (cf. [1])
\begin{equation}
\| f - \tilde{f} \|_{L^2(G)} \leq K_1 (A)^4.
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If we approximate the integrals $k_i(u)$, $i \leq M$, in (2.7) by a quadrature scheme and denote these approximations by $\tilde{k}_i(u)$, we have the following new system to solve:
\begin{equation}
Au + \tilde{k}(u) = 0.
\end{equation}

Naturally, since $f$ is dependent on $u$, we are not assured that the system (4.1) has a unique solution. As in the previous section, we will denote by $\tilde{w}_M(x, y)$ the approximation generated by the system (2.4) using subspace $S_M$.

Let $A_x: 0 = x_0 < x_1 < \cdots < x_{N_x+1} = 1$, $A_y: 0 = y_0 < y_1 < \cdots < y_{N_y+1} = 1$ be a partition of the unit square $G$ and, as in §3, denote by $A$ the set of points $(x_i, y_j)$, $0 \leq i \leq N_x + 1$, $0 \leq j \leq N_y + 1$. Again assume $A \leq \delta_0 A$, where $\delta_0$ is a fixed positive constant, and we restrict our attention to subspaces $S_M(A)$ of $S$ of piecewise-polynomial functions. Writing $k_i(u)$ in (2.7) as the sum
\begin{equation}
k_i(u) = \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} \sum_{j=1}^{M} \int_{x_l}^{x_{l+1}} \int_{y_k}^{y_{k+1}} f(x, y, \sum_{j=1}^{M} u_j w_j) w_l dy dx,
\end{equation}
and using the Cauchy-Schwarz inequality gives
\begin{equation}
\| k_i - \tilde{k}_i \|_D \leq K \| k_i - \tilde{k}_i \|_{L^2(G)}.
\end{equation}
Now $\| k_i - \tilde{k}_i \|_{L^2(G)} \leq \| k_i - \tilde{k}_i \|_{1, 2}$, and $\| \cdot \|_D$ is equivalent to $\| \cdot \|_{1, 2}$, so there exists a constant $K$ such that
\begin{equation}
\| k_i - \tilde{k}_i \|_{L^2(G)} \leq K \| k_i - \tilde{k}_i \|_D.
\end{equation}
Hence from (3.20),
\begin{equation}
\| k_i - \tilde{k}_i \|_D \leq K \| f - \tilde{f} \|_{L^2(G)}.
\end{equation}
Q.E.D.
and applying the quadrature scheme (3.7) to the \((N_x+1)(N_y+1)\) integrals in this sum, we obtain an approximation which we will denote, in simplified notation, as

\[
\tilde{k}_i(u) = \sum_{i=0}^{M_0} \sum_{k=0}^{M_1} \beta_{i,k} f \left( x_i, y_k, \sum_{j=1}^{M} u_j w_j \right) w_i(x_i, y_k). \tag{4.2}
\]

Substituting these \(\tilde{k}_i(u)\) for \(k_i(u)\) in (2.4) generates system (4.1). The following theorem gives sufficient conditions for the system of equations (4.1) to have a unique solution.

**THEOREM 3.** Suppose that \(S_M\) is a finite-dimensional subspace of \(S\) spanned by the linearly independent set \(\{w_i(x, y)\}_{i=1}^{M}\) and let the quadrature scheme (3.7), used as described above to obtain the approximations \(\tilde{k}_i(u)\) in (4.1), satisfy the following conditions:

\[
\alpha_{i,j} \geq 0, \quad 0 \leq i, j \leq m; \quad \sum_{i=0}^{M_0} \sum_{j=0}^{M_1} \alpha_{i,j} = (x_m - x_0)(y_m - y_0);
\]

and, in the notation of (4.2)

\[
\sum_{i=0}^{M_0} \sum_{k=0}^{M_1} \beta_{i,k} w_i(x_i, y_k) w_j(x_i, y_k) = \int_{0}^{1} \int_{0}^{1} w_i w_j \, dy \, dx,
\]

for \(1 \leq i, j \leq m\). Then the system (4.1) has a unique solution \(\tilde{u}\).

**Proof.** We define the following functional on \(S_M\):

\[
H[w] = \int_{0}^{1} \int_{0}^{1} \left[ \frac{\partial^2 w}{\partial x^2} \right]^2 + \left( \frac{\partial w}{\partial y} \right)^2 \, dy \, dx + \sum_{i=0}^{M_0} \sum_{k=0}^{M_1} \beta_{i,k} \int_{0}^{1} f(x_i, y_k, \eta) \, d\eta
\]

where \(w(x, y) = \sum_{j=1}^{M} u_j w_j(x, y)\). Note that the system \(0 = \partial H[w]/\partial u_i, 1 \leq i \leq M\), is exactly system (4.1) where the \(\tilde{k}_i(u), 1 \leq i \leq M\), are described in (4.2). Hence, in order to prove that (4.1) has a unique solution, it suffices to show that \(H[w]\) has a unique stationary value over \(S_M\). The remainder of the proof is simply the two-dimensional analogue of the proof of Theorem 4 in [8]. See [7, p. 157] for complete details.

**THEOREM 4.** Let \(Q\) be any collection of quasi-uniform partitions \(\Delta\) of \(G\), i.e., there exists a constant \(\delta_0 > 0\) such that \(\delta_0 \Delta \geq \Delta\) for all \(\Delta \in Q\), and for each \(\Delta \in Q\), let \(S_M(\Delta)\) be a finite-dimensional subspace of \(S\) consisting of piecewise-polynomial Hermite or spline functions of degree \(n_0\). Assume that \(\partial^m f(x, y, w(x, y)) / \partial x^m\) and \(\partial^m f(x, y, w(x, y)) / \partial y^m\) are continuous for all \(w(x, y) \in S_M(\Delta)\) for all \(\Delta \in Q\) and that \(\phi(x, y) \in C^{n_0+1}(\overline{G})\). If the quadrature scheme (3.7), used to approximate the \(k_i(u)\) in (2.7), satisfies the hypotheses of Theorem 3 for each subspace \(S_M(\Delta)\), and \(m_0 \geq n_0\), then
there exists a positive constant $K$ such that

$$
\| \tilde{w} - \hat{w} \|_r \leq K (\hat{A})^s \quad \text{for all} \quad A \in Q,
$$

where $s = \min (m_0 - n_0 - 2, n_0)$.

Proof. For any partition $A \in C$, let \{w_i(x, y)\}_{i=1}^M$ be a basis for $S_M(A)$. Denote by $\tilde{w}(x, y) = \sum_{j=1}^M \tilde{u}_j w_j$ the interpolate in $S_M$ of $\phi(x, y)$, the unique solution of (1.1). We define

$$
e_i = (A \tilde{u})_i + k_i (\tilde{u}), \quad 1 \leq i \leq M,$$

where $A$ and $k_i(\cdot)$ are defined in (2.6) and (2.7) respectively. Thus,

$$A \tilde{u} = - \tilde{k} (\tilde{u}) + \varepsilon + \tilde{k} (\tilde{u}) - k (\tilde{u}). \tag{4.3}$$

Recalling that $\tilde{u}$ satisfies

$$A \tilde{u} = - \tilde{k} (\tilde{u}), \tag{4.4}$$

subtracting (4.4) from (4.3) and premultiplying by $(\tilde{u} - \bar{u})^T$, we obtain

$$(\tilde{u} - \bar{u})^T A (\tilde{u} - \bar{u}) = (\tilde{u} - \bar{u})^T (\tilde{k} (\tilde{u}) - \tilde{k} (\bar{u})) + (\tilde{u} - \bar{u})^T (\varepsilon + \tilde{k} (\tilde{u}) - k (\tilde{u})). \tag{4.5}$$

Letting $\bar{w} = \sum_{j=1}^M \tilde{u}_j w_j$, it is easy to show from the hypotheses of Theorem 3 and (2.2), that

$$\gamma \int_0^1 \int_0^1 (\bar{w} - \tilde{w})^2 \, dy \, dx \leq (\tilde{u} - \bar{u})^T (\tilde{k} (\tilde{u}) - \tilde{k} (\bar{u})).$$

Therefore, from (4.5) and (3.5), it follows from the definition of $\| \cdot \|_r$, that

$$\| \tilde{w} - \bar{w} \|_r^2 \leq (\tilde{u} - \bar{u})^T \varepsilon + (\tilde{u} - \bar{u})^T (\tilde{k} (\tilde{u}) - k (\tilde{u})). \tag{4.6}$$

The term $(\tilde{u} - \bar{u})^T (\tilde{k} (\tilde{u}) - k (\tilde{u}))$ in (4.6) is just the error in applying the quadrature scheme (3.7) on the cells of partition $A$, to the function $f(x, y, \tilde{w}) (\tilde{w} - \bar{w})$. From (3.8) we see that this error is bounded above by

$$\sum_{i=0}^{N_x} \sum_{k=0}^{N_y} K_1 (\hat{A})^{m_0 + 2} \max_{i \in 0, m_0} \left\{ \left| \int_{x_i}^{x_{i+1}} \int_{y_k}^{y_{k+1}} f(x, y, \tilde{w}) \, dx \, dy \right| \right\}_{L^\infty},$$

where

$$\| \cdot \|_{L^\infty} = \| \cdot \|_{L^\infty ((x_i, x_{i+1}) \times (y_k, y_{k+1}))}.$$

Then, using the Markov theorem as in the proof of Theorem 1, the assumed continuity properties of the function $f$ and the fact that derivatives of the Hermite or spline interpolate $\tilde{w}$ are bounded ([2] and [1]) since we are assuming that $\phi(x, y) \in C^{m_0 + 1} (\hat{G})$, we see that there exist constants $K_{i, k}$, such that

$$(\tilde{u} - \bar{u})^T (k (\tilde{u}) - \tilde{k} (\tilde{u})) \leq \sum_{i=0}^{N_x} \sum_{k=0}^{N_y} K_{i, k} (\hat{A})^{m_0 + 2 - n_0} \| \tilde{w} - \bar{w} \|_{L^\infty (G)}.$$
Therefore, by the same argument as in §3, there is a constant $K_1$ such that
\[ |(\mathbf{\bar{u}} - \mathbf{\bar{u}})^T (k(\mathbf{\bar{u}}) - \mathbf{\bar{k}}(\mathbf{\bar{u}}))| \leq K_1 (N_x + 1) (N_y + 1) (\mathbf{\bar{A}})^{m_0 - n_0} \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_{L^2(G)}.\]

Now, since $A \max_{1,2} (N_x + 1, N_y + 1) (A) \leq \delta_0 A$, then
\[ K_1 (N_x + 1) (N_y + 1) (\mathbf{\bar{A}})^{m_0 - n_0} \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_{L^2(G)} \leq K_1 \delta_0^2 (A)^{m_0 - n_0 - 2} \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_{L^2(G)}.\]

As noted in the previous section, $\|\cdot\|_{L^2} \ll \|\cdot\|_{1,2}$ and $\|\cdot\|_{1,2}$ is equivalent to $\|\cdot\|_y$, so there is a constant $K_2$ such that
\[ |(\mathbf{\bar{u}} - \mathbf{\bar{u}})^T (k(\mathbf{\bar{u}}) - \mathbf{\bar{k}}(\mathbf{\bar{u}}))| \leq K_2 (\mathbf{\bar{A}})^{m_0 - n_0 - 2} \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y. \quad (4.7)\]

Combining (4.6) and (4.7), we have
\[ \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y^2 \leq |(\mathbf{\bar{u}} - \mathbf{\bar{u}})^T \mathbf{e}| + K_2 (\mathbf{\bar{A}})^{m_0 - n_0 - 2} \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y. \quad (4.8)\]

As in one dimension ([8, §4] and [7, pp. 62–66]), it can be verified that there exists a constant $K_3$ such that
\[ |(\mathbf{\bar{u}} - \mathbf{\bar{u}})^T \mathbf{e}| \leq K_3 \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y. \]

Therefore, (4.8) becomes
\[ \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y \leq K_3 \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y + K_2 (\mathbf{\bar{A}})^{m_0 - n_0 - 2}, \]
and hence
\[ \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y \leq \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y + \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y \]
\[ \leq (1 + K_3) \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y + K_2 (\mathbf{\bar{A}})^{m_0 - n_0 - 2} \]
\[ \leq (1 + K_3) \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y + \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y + K_2 (\mathbf{\bar{A}})^{m_0 - n_0 - 2}. \]

However, $\|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y \leq C \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y$, from (2.13), so that
\[ \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y \leq (1 + K_3) (1 + C) \|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y + K_2 (\mathbf{\bar{A}})^{m_0 - n_0 - 2}. \]

Then, as $\mathbf{\bar{w}}$ is the interpolation of $\mathbf{\bar{w}}$ in $S_M$, the error bounds of $[2]$ give us that $\|\mathbf{\bar{w}} - \mathbf{\bar{w}}\|_y \leq K(\mathbf{\bar{A}})^n$, from which the desired conclusion of Theorem 4 follows.

Q.E.D.

This brings us to the analogue of Definition 1 for the nonlinear problem.

**DEFINITION 2.** Let $Q$ be any collection of quasi-uniform partitions of $G$, and for each $\Delta \in Q$, let $S_M(\Delta)$ be a finite-dimensional Hermite or spline subspace of $S$. Let $\mathbf{\bar{w}}_M$, the function which minimizes $F[\mathbf{\bar{w}}]$ of (2.3) over $S_M(\Delta)$, satisfy
\[ \|\mathbf{\bar{w}}_M - \phi\|_N \leq K(\mathbf{\bar{A}})^l \text{ for all } \Delta \in Q, \quad (4.9)\]

where $K$ and $l$ are positive constants independent of $\Delta$, $\phi$ is the solution of (1.1), and $\|\cdot\|_N$ is some norm on $S$. Then, the choice of quadrature schemes in (3.7) is consistent in the norm $\|\cdot\|_N$ with the bounds of (4.9) if there exists a positive constant $K_0$, independent of $\Delta$, such that
\[ \|\mathbf{\bar{w}}_M - \mathbf{\bar{w}}_M\|_N \leq K_0 (\mathbf{\bar{A}})^l \text{ for all } \Delta \in Q. \]
COROLLARY. If the hypotheses of Theorem 4 hold, then the quadrature scheme (3.7) is consistent in the norm \( \| \cdot \|_\gamma \) with the bounds \( \| w_M - \phi \| \leq K_4 (\lambda)^m \) deduced from [2] and [1], if \( m_0 \geq 2n_0 + 2 \).

When \( f \) in (1.1) is a function of \( u \) as well as \( x \) and \( y \), approximating the integrals \( k_i (u) \) in (2.7) by interpolating \( f (x, y, \sum_{j=1}^M u_j w_j) \) by some \( \tilde{f} (x, y, \sum_{j=1}^M u_j w_j) \) and then evaluating

\[
\int \int_G \tilde{f} (x, y, \sum_{j=1}^M u_j w_j) w_i \, dy \, dx
\]

effectively, is not advantageous. The reason, as shown in [8, §4], is that the system of equations which is generated cannot be shown to be the gradient of some functional set to zero and hence, we cannot be sure that the attractive techniques available for minimizing functionals are applicable.

We should mention here that if we restrict ourselves to piecewise Hermite interpolation subspaces, then these results may be generalized from rectangles to rectangular polygons, as in [2].

5. Numerical Examples

We now discuss some examples of problems of the form (1.1) whose solutions have been approximated by the techniques discussed in the previous sections.

Consider the problem

\[
\begin{aligned}
Au (x, y) &= 6xye^{x^3} (xy + x + y - 3), \quad (x, y) \in G, \\
u (x, y) &= 0, \quad (x, y) \in \partial G,
\end{aligned}
\]

(5.1)

where \( G \) is the open unit square \((0, 1) \times (0, 1)\). The quantity \( A_1 \) of (2.1) in this case is \( 2\pi^2 \), as noted in [6, p. 249]. For this problem, \( \gamma \) in (2.2) can be chosen to be zero.

Since \( 6xye^{x^3} (xy + x + y - 3) \) is independent of \( u \), then (2.9) holds with \( K = 0 \). The unique solution to (5.1) is

\[
u (x, y) = 3e^{x^3} (x - x^2) (y - y^2).
\]

The first way that the solution of (5.1) was approximated was by minimizing the functional

\[
F[w] = \int_G \left\{ \frac{1}{2} \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] + 6xye^{x^3} (xy + x + y - 3) w(x, y) \right\} \, dy \, dx
\]

(5.2)

over the piecewise linear Hermite subspace \( H^1_0 (\pi_N) \) of \( S \), described in [2, §4], where \( \pi_N \) is the uniform mesh on \( G \) with mesh size \( h_N = 1/(N+1) \). The dimension of \( H^1_0 (\pi_N) \) is \( N^2 \). Denoting the basis functions of \( H^1 (\pi_N) \) as \( \{ w_i (x, y) \} \) for \( i = 1, \ldots, N^2 \), let

\[
\sum_{j=1}^{N^2} \tilde{w}_j w_j (x, y) = \tilde{w}_N (x, y)
\]

be the unique minimum of \( F[w] \) in (5.2) over \( H^1_0 (\pi_N) \).
By [2, Theorem 16] and the fact that \( \| \cdot \|_{1,2} \) is equivalent to \( \| \cdot \|_D \) we know that

\[
\| \phi - \tilde{\omega}_N \|_D \leq K h_N, \quad N \geq 1, \tag{5.3}
\]

where \( K \) is a constant independent of \( N \). The two-dimensional four-point Gaussian quadrature scheme generated as in (3.7) from the one-dimensional two-point Gaussian quadrature scheme with weight function unity, was used to approximate the integrals of the form (3.1). If we denote by \( \tilde{\omega}_N(x, y) \) the approximation resulting from the minimization of \( F[w] \) using the above mentioned quadrature scheme, then by Theorem 1, with \( m_0 = 4 \) and \( n_0 = 1 \), we can easily verify that there is a constant \( K' \), independent of \( N \), such that

\[
\| \tilde{\omega}_N - \tilde{\omega}_N \|_D \leq K' h_N.
\]

Therefore, by Definition 1, the two-dimensional four-point Gaussian quadrature scheme is consistent in the norm \( \| \cdot \|_D \) with the bounds of (5.3). Consequently, we have

\[
\| \phi - \tilde{\omega}_N \|_D \leq K h_N, \quad N \geq 1,
\]

where \( K \) is a constant independent of \( N \). The numerical results are given in Table 1. The quantity \( \alpha \) in this table is

\[
\alpha \equiv \log \left( \frac{\| \phi - \tilde{\omega}_{n_1} \|_{L^\infty(G)}}{\| \phi - \tilde{\omega}_{n_2} \|_{L^\infty(G)}} \right) \log \left( \frac{h_{n_1}}{h_{n_2}} \right), \tag{5.4}
\]

which is an estimate of the rate of convergence of our approximations, defined in terms of successive values of the mesh spacing \( h \). Equation (5.4) is derived from the fact that as \( h_N \to 0 \), we have

\[
\| \phi - \tilde{\omega}_N \|_{L^\infty(G)} \sim K (h_N)^\alpha
\]

for some constants \( \alpha \) and \( K \) which are independent of \( h_N \). Then, for two successive values of \( h, h_{n_1} > h_{n_2} \), we have asymptotically

\[
\frac{\| \phi - \tilde{\omega}_{n_1} \|_{L^\infty(G)}}{\| \phi - \tilde{\omega}_{n_2} \|_{L^\infty(G)}} \sim \left( \frac{h_{n_1}}{h_{n_2}} \right)^\alpha, \tag{5.5}
\]

and (5.4) follows from (5.5). We see from Table 1 that the accuracy seems to be \( O(h_N^2) \) in the norm \( \| \cdot \|_{L^\infty(G)} \).

<table>
<thead>
<tr>
<th>( h_N )</th>
<th>dimension of ( H_0^1(\pi_N) )</th>
<th>( | \phi - \tilde{\omega}<em>N |</em>{L^\infty(G)} )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/7</td>
<td>36</td>
<td>3.10 \cdot 10^{-2}</td>
<td>-</td>
</tr>
<tr>
<td>1/8</td>
<td>49</td>
<td>2.43 \cdot 10^{-2}</td>
<td>1.84</td>
</tr>
<tr>
<td>1/9</td>
<td>64</td>
<td>1.96 \cdot 10^{-2}</td>
<td>1.85</td>
</tr>
<tr>
<td>1/10</td>
<td>81</td>
<td>1.60 \cdot 10^{-2}</td>
<td>1.94</td>
</tr>
<tr>
<td>1/11</td>
<td>100</td>
<td>1.33 \cdot 10^{-2}</td>
<td>1.97</td>
</tr>
</tbody>
</table>
The solution of problem (5.1) was also approximated by minimizing $F[w]$ in (5.2) over the piecewise cubic Hermite subspace $H_0^3(\pi_N)$ of $S$ ([2, 4]), where $\pi_N$ is the same as above. The dimension of $H_0^3(\pi_N)$ is $4N^2 + 8N + 4$, and if we denote by $\hat{w}_N(x, y)$ the element of $H_0^3(\pi_N)$ which uniquely minimizes $F[w]$ over $H_0^3(\pi_N)$, then, again using [2, Theorem 16], we can deduce that

$$\|\phi - \hat{w}_N\|_D \leq K h_N^3, \quad N \geq 1,$$

(5.6)

for some constant $K$, independent of $N$. For the piecewise cubic polynomial subspace $H_0^3(\pi_N)$, the two-dimensional sixteen-point Gaussian quadrature scheme derived from the one-dimensional fourpoint Gaussian quadrature scheme with weight function unity, was used to approximate the integrals in the system generated by minimizing $F[w]$. For this quadrature scheme, $m_o = 8$ and by Theorem 1, with $n_0 = 3$, we can verify that

$$\|\hat{w}_N - \tilde{w}_N\|_D \leq K h_N^3, \quad N \geq 1,$$

where $\tilde{w}_N$ is the minimum of $F[w]$ over $H_0^3(\pi_N)$ using the quadrature scheme. Hence, by Definition 1, this Gaussian quadrature scheme is consistent in the norm $\|\cdot\|_D$ with (5.6) and we have

$$\|\phi - \hat{w}_N\|_D \leq K h_N^3, \quad N \geq 1,$$

where $K$ is a constant independent of $N$. The numerical results are given in Table 2.

### Table 2

<table>
<thead>
<tr>
<th>$h_N$</th>
<th>Dimension of $H_0^3(\pi_N)$</th>
<th>$|\phi - \hat{w}<em>N|</em>{L^\infty(\Omega)}$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>36</td>
<td>$9.11 \cdot 10^{-4}$</td>
<td>-</td>
</tr>
<tr>
<td>1/4</td>
<td>64</td>
<td>$3.15 \cdot 10^{-4}$</td>
<td>3.70</td>
</tr>
<tr>
<td>1/5</td>
<td>100</td>
<td>$1.32 \cdot 10^{-4}$</td>
<td>3.92</td>
</tr>
<tr>
<td>1/6</td>
<td>144</td>
<td>$7.06 \cdot 10^{-5}$</td>
<td>3.93</td>
</tr>
</tbody>
</table>

The $\alpha$ values in this table indicate that for the subspace $H_0^3(\pi_N)$ the accuracy in the norm $\|\cdot\|_{L^\infty(\Omega)}$ is probably $O(h_N^4)$.

The final subspace over which we minimized $F[w]$ in (5.2) was the piecewise-cubic-polynomial spline ([1]) subspace $Sp_0^3(\pi_N)$ of $S$, where $\pi_N$ is once again the uniform mesh on the unit square. From [6], we can easily deduce that if $\tilde{w}_N(x, y)$ is the minimum of $F[w]$ over $Sp_0^3(\pi_N)$, then

$$\|\phi - \hat{w}_N\|_D \leq K h_N^3, \quad N \geq 1.$$

(5.7)

The dimension of $Sp_0^3(\pi_N)$ is $N^2 + 4N + 4$ and the two-dimensional, sixteen-point Gaussian quadrature scheme was used to approximate the integrals involved in
minimizing $F[w]$. Letting $\tilde{w}_N$ denote the approximations resulting from minimizing $F[w]$ over $S\rho_0^2(\pi_N)$ using the quadrature scheme, by Theorem 1 we see that
\[ \|\tilde{w}_N - \hat{w}_N\|_D \leq K h_N^3, \quad N \geq 1, \]
and hence by Definition 1, the quadrature scheme is consistent in $\|\cdot\|_D$ with (5.7). Consequently
\[ \|\phi - \hat{w}_N\|_D \leq K h_N^3, \quad N \geq 1, \]
where $K$ is a constant independent of $N$. The numerical results are given in Table 3. They indicate that the accuracy in the norm $\|\cdot\|_{L^\infty(G)}$ is $O(h_N^4)$.

<table>
<thead>
<tr>
<th>$h_N$</th>
<th>dimension of $S\rho_0^2(\pi_N)$</th>
<th>$|\phi - \hat{w}<em>N|</em>{L^\infty(G)}$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>16</td>
<td>1.08·10^{-3}</td>
<td>-</td>
</tr>
<tr>
<td>1/4</td>
<td>25</td>
<td>3.57·10^{-4}</td>
<td>3.84</td>
</tr>
<tr>
<td>1/5</td>
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<td>1.53·10^{-4}</td>
<td>3.84</td>
</tr>
<tr>
<td>1/6</td>
<td>49</td>
<td>7.66·10^{-5}</td>
<td>3.82</td>
</tr>
<tr>
<td>1/7</td>
<td>64</td>
<td>4.19·10^{-6}</td>
<td>3.94</td>
</tr>
</tbody>
</table>

As our second and final example, we consider
\begin{align*}
\Delta u(x, y) &= (u)^3 + (-2 + (1 - 2x)^2) (e^{y(1-y)} - 1 + u) \\
&\quad + (-2 + (1 - 2y)^2) (e^{x(1-x)} - 1 + u) \\
&\quad - (e^{x(1-x)} - 1)^3 (e^{y(1-y)} - 1)^3, \quad (x, y) \in G, \tag{5.8}
\end{align*}
\begin{align*}
u(x, y) &= 0, \quad (x, y) \in \partial G, \tag{5.9}
\end{align*}

where $G$ is the open unit square. As in problem (5.1), $A_1$ in (2.1) is $2\pi^2$ and $y$ in (2.2) can be set equal to $-4$. Denoting the right side of (5.8) by $f(x, y, u)$, $\partial f/\partial u$ is in $C^0(G \times R)$, but it is unbounded as $|u| \to +\infty$. We now show that (2.11) is satisfied. We first estimate the quantity $-\varrho^{-1}$, where $\varrho$ is defined as follows. Let $\Psi(x, y)$ be the unique solution of
\begin{align*}
\Delta u(x, y) &= 1, \quad (x, y) \in G, \\
u(x, y) &= 0, \quad (x, y) \in \partial G, \tag{5.10}
\end{align*}

where $G$ is the open unit unit square. Then
\[ \varrho = \sup_{(x, y) \in G} |\Psi(x, y)| > 0. \]

To estimate $\varrho$, we use the following maximum principle [14, p. 56]. If $G'$ is a domain...
in the \((x, y)\) plane with boundary \(\partial G'\) and \(\phi(x, y)\) is a solution to the problem

\[
\begin{align*}
\Delta u(x, y) &= -F(x, y), \quad (x, y) \in G' \\
u(x, y) &= f(x, y), \quad (x, y) \in \partial G',
\end{align*}
\]

where \(F(x, y) < 0\) for \((x, y) \in G'\), then

\[
\max_{(x, y) \in G'} \vert \phi(x, y) \vert \leq \max_{\partial G'} \vert f \vert + \frac{1}{4} R^2 \max_{G'} \vert F \vert
\]

where \(R\) is the radius of a circle containing \(G'\). For our problem (5.10), \(f = 0\), \(F \equiv -1\), and we may let \(R = \sqrt{2}/2\). Therefore,

\[
\rho = \sup_{(x, y) \in G} \vert \psi(x, y) \vert \leq \frac{1}{4} \left( \frac{1}{2} \right)^2 = \frac{1}{8},
\]

so that \(-\rho^{-1} \leq -8\). Since \(\gamma = -4\) and \(-\rho^{-1} \leq -8\), then (2.11) is satisfied for the problem (5.8)–(5.9). The unique solution to (5.8)–(5.9) is

\[
u(x, y) = (e^{x(1-x)} - 1) (e^{y(1-y)} - 1).
\]

In approximating the solution of (5.8)–(5.9) by the variational method, the function which we minimize is

\[
F[w] = \iint_G \left\{ \frac{1}{2} \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] + \int_0^w f(x, y, \eta) \, d\eta \right\} \, dy \, dx,
\]

(5.11)

where \(f(x, y, u)\) represents the right side of (5.8).

The first subspace over which we minimized \(F[w]\) in (5.11) was the piecewise linear Hermite subspace \(H^1_0(\pi_N)\), where \(\pi_N\) is again the uniform mesh on the unit square with mesh size \(h_N = 1/(N+1)\). If we denote by \(\hat{\nu}_N(x, y)\) the minimum of \(F[w]\) over \(H^1_0(\pi_N)\), then we can deduce from [2, Theorem 16] that there is a constant \(K\) such that

\[
\|u - \hat{\nu}_N\|_y \leq Kh_N, \quad N \geq 1.
\]

(5.12)

By Theorem 3 and Theorem 4 we see that when the two-dimensional, four-point Gaussian quadrature scheme is used to approximate the integrals arising in the minimization of \(F[w]\), the resulting approximation \(\hat{\nu}_N(x, y)\) satisfies

\[
\|\hat{\nu}_N - \hat{\nu}_N\|_y \leq Kh_N, \quad N \geq 1.
\]

Hence, by Definition 2, this quadrature scheme is consistent in the norm \(\| \cdot \|_y\) with bounds (5.12). The numerical results are given in Table 4.

The subspaces \(H^3_0(\pi_N)\) of piecewise cubic Hermite polynomials and \(Sp^3_0(\pi_N)\) of piecewise cubic spline polynomials were also used to approximate the solution of (5.8)–(5.9). By [2, Theorem 16] and [1], these subspaces yield, through the minimization of \(F[w]\), approximations which are \(O(h_N)\) in the norm \(\| \cdot \|_y\). Next, by Theorems
Table 4

<table>
<thead>
<tr>
<th>( h_N )</th>
<th>dimension of ( H_0^3(\pi_N) )</th>
<th>( | u - \tilde{w}<em>N |</em>{L^\infty(G)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>9</td>
<td>7.22 \times 10^{-3}</td>
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<tr>
<td>1/5</td>
<td>16</td>
<td>5.22 \times 10^{-3}</td>
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<tr>
<td>1/6</td>
<td>25</td>
<td>3.46 \times 10^{-3}</td>
</tr>
<tr>
<td>1/7</td>
<td>36</td>
<td>2.69 \times 10^{-3}</td>
</tr>
<tr>
<td>1/8</td>
<td>49</td>
<td>2.01 \times 10^{-3}</td>
</tr>
</tbody>
</table>

3 and 4, we can verify that the two-dimensional, sixteen-point Gaussian quadrature scheme is consistent by Definition 2 in the norm \( \| \cdot \|_\gamma \) with these \( O(h_N^2) \) bounds. Letting \( \tilde{w}_N(\mathbf{x}, \mathbf{y}) \) denote the approximations generated by minimizing \( F[w] \) over the Hermite and spline subspaces and using the above-mentioned quadrature scheme, the numerical results are given in Tables 5 and 6.

Table 5

<table>
<thead>
<tr>
<th>( h_N )</th>
<th>dimension of ( H_0^2(\pi_N) )</th>
<th>( | u - \tilde{w}<em>N |</em>{L^\infty(G)} )</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>1/3</td>
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</tr>
<tr>
<td>1/4</td>
<td>64</td>
<td>4.06 \times 10^{-5}</td>
</tr>
<tr>
<td>1/5</td>
<td>100</td>
<td>8.67 \times 10^{-6}</td>
</tr>
<tr>
<td>1/6</td>
<td>144</td>
<td>5.28 \times 10^{-6}</td>
</tr>
</tbody>
</table>

Table 6

<table>
<thead>
<tr>
<th>( h_N )</th>
<th>dimension of ( S_0^2(\pi_N) )</th>
<th>( | u - \tilde{w}<em>N |</em>{L^\infty(G)} )</th>
</tr>
</thead>
<tbody>
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</tr>
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<td>1/4</td>
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<td>9.76 \times 10^{-6}</td>
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REFERENCES


[7] HERBOLD, R. J., Consistent Quadrature Schemes for the Numerical Solution of Boundary Value Problems by Variational Techniques (Doctoral Thesis, Case Western Reserve University, 1968 (189 pp.)).