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Let $A = [a_{i,j}]$ be an $n \times n$ matrix consisting of non-negative elements. It is well known [1, p. 463] that $A$ is primitive if and only if, for some positive integer $n$, $A^n$ has all its elements positive. One needs to know only this property of primitive matrices to understand this paper. If $A^k$ is positive (i.e. has all its elements positive), then $A^h$ is also positive for all integers $h > k$ [1, p. 463]. Letting $A$ be primitive, we shall define $\gamma(A)$ as the smallest positive integer $h$ such that $A^h$ is positive.

Wielandt [2, p. 648] stated without proof the inequality

\begin{equation}
\gamma(A) \leq n^2 - 2n + 2,
\end{equation}

and gave an example to show that $\gamma(A)$ could equal $n^2 - 2n + 2$. In the special case that all the diagonal elements of $A$ are positive, Wielandt [2, p. 644] showed that one may obtain the better bound

\begin{equation}
\gamma(A) \leq n - 1.
\end{equation}

In this paper, we show that when there are one or more positive diagonal elements of $A$ (or of one of its low order powers), bounds may be found for $\gamma(A)$ which are better than (1), although not necessarily as good as (2). We shall also give an easy proof of (1).

In our discussion, we shall assume that the matrix $A$ is non-negative and primitive. Let $J$ be the set of positive integers one through $n$. For $L$ a subset of $J$, define $F^0(L) = L$ and, by induction, for $k$ a positive integer, define $F^k(L)$ as the set of all $i \in J$ such that for some $j \in F^{k-1}(L)$, $a_{i,j} > 0$. For $h$ a non-negative integer, and $j \in J$, define $F^h(j)$ as $F^h(L)$ where $L$ is the set containing $j$ and only $j$. We remark that, for $h$ a positive integer, the element of $A^h$ in the $i$th row and $j$th column is positive if and only if $i \in F^h(j)$.

**Lemma 1.** $F(J) = J$.

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1 Work done under the auspices of the A.E.C.

2 One may also use Lemma 1 of this paper.

3 Others, in examining the fundamental properties of non-negative primitive matrices have indirectly obtained bounds for $\gamma(A)$. For example, as pointed out by Wielandt [2, p. 647], Frobenius [1, p. 463] indirectly obtained the bound $2n^2 - 2n$, while Herstein [3, p. 20] indirectly obtained the bound $n^2$ for $\gamma(A)$.

4 This obviously implies that $A$ is irreducible. See [1, p. 463].

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Proof. For \( j \in J, J = F_{r(A)}(j) \subseteq F_{r(A)}(J) = F[F_{r(A)}^{-1}(J)] \subseteq F(J) \subseteq J \).

Lemma 2. If \( L \) is a proper subset of \( J \), then \( F(L) \) contains some element not in \( L \).

Proof. If not, then \( J \supseteq L \supseteq F(L) \supseteq \ldots \supseteq F_{r(A)}(L) = J \) which contradicts \( J \neq L \).

Corollary. If \( h \leq n - 1 \), then \( \{j\} \cup F(j) \cup \ldots \cup F^h(j) \) contains at least \( h + 1 \) elements.

Proof. This is obviously true for \( h = 0 \). Using mathematical induction, assume it is true for some \( 0 \leq h \leq n - 1 \). Set \( L = \{j\} \cup \ldots \cup F^h(j) \), and apply Lemma 2.

We remark that, given \( j \in J \), the set of integers \( h \) such that \( j \in F^h(j) \) is a semigroup. Therefore, properties described below may be easily observed by observing the first few iterates of \( A \).

Lemma 3. Let \( k \) be a non-negative integer, and \( j \in J \). For \( h \geq k \), let \( j \in F^h(j) \). Then, \( F^{n-h+k}(j) = J \).

Proof. The corollary above implies that \( \{j\} \cup \ldots \cup F^{n-1}(j) = J \). For each \( 0 \leq h \leq n - 1 \), \( j \in F^{n-1+h-j}(j) \), and so \( F^h(j) \subseteq F^{n-1+k}(j) \). Therefore, \( J = \bigcup_{h=0}^{n-1} F^h(j) \subseteq F^{n-1+k}(j) \subseteq J \).

Theorem 1. Let \( k \) be a non-negative integer. Let there be at least \( d > 0 \) elements \( j \) of \( J \) such that for \( h \geq k \), the \( j \)-th diagonal element of \( A^k \) is positive. Then, \( \gamma(A) \leq 2n - d - 1 \).

Proof. The corollary above implies that, for each \( j \in J \), there exists \( 0 \leq h \leq n - d \) such that \( F^h(j) \) contains at least one of the \( d \) elements described above. Then,

\[
J \supseteq F^{2n-d-1+h}(j) = F^{n-d-h} \{F^{n-1+h}(j)\} \supseteq F^{n-d-h}(J) = J.
\]

Corollary. Let at least \( d > 0 \) of the diagonal elements of \( A \) be positive. Then, \( \gamma(A) \leq 2n - d - 1 \).

Theorem 2. Let \( h \) be a positive integer, and let \( A + A^2 + \ldots + A^h \) have at least \( d > 0 \) of its diagonal elements positive. Then, \( \gamma(A) \leq n - d + h(n-1) \).

Proof. Let \( j \) be one of the \( d \) elements such that \( j \in F^p(j) \) for some \( p, 1 \leq p \leq h \). Then, if we substitute 0 for \( k \), and \( F^p \) for \( F \), we may apply Lemma 3, and conclude that \( F^{(n-1)p}(j) = J \). Choose arbitrarily \( j' \in J \). Then, the corollary to Lemma 2 implies that there exists an \( l \),

\[^6\text{If all the diagonal elements of } A \text{ are positive, then } d = n, \text{ and the inequality of the corollary reduces to Wielandt's result (2).} \]
0 \leq l \leq n - d \text{ such that } F^l(j') \text{ contains at least one of these } d \text{ elements. Therefore, } J \supseteq F^{n-d+k(n-1)}(j') = F^{n-d-l+(h-p)(n-1)} \left\{ F^{l-1}(j') \right\} \supseteq F^{n-d-l+(h-p)(n-1)}(J) = J, \text{ since } n - d - l + (h-p)(n-1) \geq 0.

Corollary. Let } A \text{ be non-negative and positively symmetric in that } a_{i,i} > 0 \text{ if and only if } a_{i,j} > 0. \text{ Then, } \gamma(A) \leq 2(n-1).

Proof. } A^2 \text{ has all its diagonal elements positive. Now, apply Theorem 2.}

Theorem 3. } \gamma(A) \leq n^2 - 2n + 2.

Proof. Given } j \in J, \text{ consider the case where } \{ j \} \cup \cdots \cup F^{n-2}(j) \neq J. \text{ Then, for } 1 \leq h \leq n - 1, F^h(j) \text{ contains exactly one element not in } \{ j \} \cup \cdots \cup F^{h-1}(j). \text{ Let } p \text{ be the smallest positive integer such that } F^p(j) \text{ contains at least two elements. Then, there exists an integer } m < p \text{ such that } m > 0 \text{ (unless } p = 1, \text{ in which case } m = 0) \text{ and such that } F^m(j) \subseteq F^p(j) = F^{m+(p-m)}(j) \subseteq F^{m+2(p-m)}(j) \subseteq \cdots. \text{ Lemma 2 implies that } F^{n-(n-1)}(p-m)(j) = J. \text{ But } p \leq n \text{ implies that } m + (n-1)(p-m) = p + (n-2)(p-m) \leq n^2 - 2n + 2.

If } \{ j \} \cup \cdots \cup F^{n-2}(j) = J, \text{ then there exists an integer } h, 0 \leq h \leq n - 1, \text{ such that } F^h(j) \subseteq F^h(j) \subseteq \cdots \subseteq F^{(n-1)h}(j) = J. \text{ But, } (n-1)h \leq n^2 - 2n + 1 < n^2 - 2n + 2. \text{ This completes the proof.}

Let } A \text{ and } B \text{ be two non-negative primitive matrices such that if } A = \| a_{i,j} \|, \text{ and } B = \| b_{i,j} \|, \text{ then } a_{i,j} > 0 \text{ implies that } b_{i,j} > 0. \text{ It is clear that } \gamma(A) \geq \gamma(B). \text{ Furthermore, if } B \text{ has many positive elements for which there are no corresponding positive elements of } A, \text{ then one would expect to have } \gamma(A) > \gamma(B). \text{ We shall show that when there are many positive off-diagonal elements of a non-negative primitive matrix, some of the preceding inequalities may be improved.}

Given a positive integer } j, 1 \leq j \leq n, \text{ define } X(j) \text{ as the number of elements } a_{i,j}, i \neq j, \text{ for which } a_{i,j} > 0. \text{ Then, the corollary to Lemma 2 implies that } X(j) \geq 1 \text{ whenever } n > 1, \text{ for all } j. \text{ Whenever } X(j) > 1, \text{ we may improve the result of the corollary to Lemma 2 by observing that if } 1 \leq h \leq n - X(j), \text{ then } \{ j \} \cup F(j) \cup \cdots \cup F^h(j) \text{ contains at least } h + X(j) \text{ elements. If we use this result in the proofs of Lemma 3 and Theorem 1, we obtain the following improvements.}

Lemma 4. Let } k \text{ and } j \text{ be as in Lemma 3. Then, } F^{n-X(j)+k}(j) = J.

Theorem 4. Let } A \text{ be as in Theorem 1. Let } X_1 \text{ be the minimum of } X(j) \text{ for the } d \text{ elements } j \in J. \text{ Let } X_2 \text{ be the minimum of } X(j) \text{ for the remaining } n-d \text{ elements } j \in J. \text{ Then,}

\[ \gamma(A) \leq 2n - d - X_1 - \min \left[ X_2 - 1; n - d \right] + k. \]
Corollary. Let \( d > 0 \) of the diagonal elements of \( A \) be positive. Then,
\[
\gamma(A) \leq 2n - d - X_1 - \min [X_2 - 1; n - d].
\]
A similar improvement may also be obtained for Theorem 2.

For any non-negative irreducible matrix, we may define the (irreducible) order of \( A \), denoted by \( \Lambda(A) \), as the smallest positive integer \( h \) such that \( I + A + A^2 + \cdots + A^h \) is positive, or equivalently, \( \{j\} \cup \cdots \cup F^h(j) = J \) for each \( j \). By definition of irreducibility, it is clear that \( \Lambda(A) \leq n - 1 \). If \( \Lambda(A) \) is less than \( n - 1 \), and the value of \( \Lambda(A) \) is known, many of the preceding inequalities may be improved. We summarize how the order of \( A \) may be used to sharpen respectively the results of Lemma 4, Theorem 4, and its corollary above. These results are respectively:

(3) \( F_{\min \{n-X(j); \Lambda(A)\}}(j) = J \),

(4) \[
\gamma(A) \leq \min [n - X_1; \Lambda(A)] + \min \{n - d - \min [X_2 - 1; n - d]; \Lambda(A)\} + k,
\]

(5) \[
\gamma(A) \leq \min [n - X_1; \Lambda(A)] + \min \{n - d - \min [X_2 - 1; n - d]; \Lambda(A)\}.
\]

Bibliography


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