CONVERSE THEOREMS AND EXTENSIONS IN
CHEBYSHEV RATIONAL APPROXIMATION TO
CERTAIN ENTIRE FUNCTIONS IN \([0, +\infty)\)

BY

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ABSTRACT. Recent interest in rational approximations to \(e^{-x}\) in \([0, + \infty)\),
arising naturally in numerical methods for approximating solutions of heat-conduction-type
parabolic differential equations, has generated results showing that
the best Chebyshev rational approximations to \(e^{-x}\), and to reciprocals of certain
entire functions, have errors for the interval \([0, + \infty)\) which converge geometrically
to zero. We present here some related converse results in the spirit of the
work of S. N. Bernstein.

1. Introduction. In numerical methods for approximating solutions of heat-
conduction-type equations, one necessarily considers matrix approximations of
\(e^{-tA}\),

\[(1.1) \quad \lambda_{m,n} = \inf_{\pi_{m,n}} \| e^{-x} - r_{m,n}(x) \|_{L_{\infty}[0,\infty]} \quad 0 \leq m \leq n, \]

where \(A\) is an \(n \times n\) Hermitian and positive definite matrix, and where \(t\), the
time parameter, belongs to \([0, + \infty)\). Though rational Padé matrix approximations
of \(e^{-tA}\) are familiar (cf. [7, Chapter 8]), rational Chebyshev matrix approximations
of \(e^{-tA}\) have only recently been considered ([2], [8]), and are associated with
rational approximations of \(e^{-x}\) in \([0, + \infty)\). Specifically, if for any nonnegative
integer \(m\), \(\pi_m\) denotes the collection of real polynomials of degree at most \(m,\)
and if \(\pi_{m,n}\) for any nonnegative integers \(m\) and \(n\) similarly denotes the collection
of all real rational functions \(r_{m,n}(x) = p_m(x)/q_n(x),\) \(p_m \in \pi_m, q_n \in \pi_n,\)
then let

\[(1.2) \quad \lambda_{m,n} = \inf_{\pi_{m,n}} \| e^{-x} - r_{m,n}(x) \|_{L_{\infty}[0,\infty]} \quad 0 \leq m \leq n, \]

denote the Chebyshev constants for \(e^{-x}\) on \([0, + \infty)\).

In [2], the following was established.

Theorem 1. Let \(\{m(n)\}_{n=0}^{\infty}\) be any sequence of nonnegative integers with
\(0 \leq m(n) \leq n\) for each \(n \geq 0\). Then the Chebyshev constants \(\lambda_{m(n),n}\) for \(e^{-x}\) in
\((0, + \infty)\) converge geometrically to zero, i.e.

\[
\lim_{n \to \infty} (\lambda_{m(n), n})^{1/n} \leq \frac{1}{2^\alpha} \leq \frac{1}{2.293...} < 1
\]

where \(\alpha = 0.1392...\) is the real solution of \(2^\alpha e^{2\alpha + 1} = 1\). Moreover,

\[
\lim_{n \to \infty} (\lambda_{0, n})^{1/n} \geq \frac{1}{6}.
\]

In [4], the geometric convergence of the Chebyshev constants to zero was extended to a wider class of entire functions. Let \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) be an entire function with \(M_f(r) = \sup_{|z|=r} |f(z)|\) its maximum modulus function. Then, \(f\) is said to be of perfectly regular growth \((\rho, B)\) (cf. Boas [1, p. 8] and Valiron [6, p. 45]) iff there exist two (finite) positive numbers \(\rho\) and \(B\) such that

\[
\lim_{r \to \infty} \frac{\ln M_f(r)}{r^\rho} = B.
\]

In [4], the following was established.

**Theorem 2.** Let \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) be any entire function of perfectly regular growth \((\rho, B)\) with \(a_k \geq 0\) for all \(k \geq 0\), and for any nonnegative integers \(m\) and \(n\) with \(0 \leq m \leq n\), let

\[
\lambda_{m, n} = \inf_{r_{m, n}} \left\{ \frac{1}{f(x)} = r_{m, n}(x) \right\} \in [0, \infty]
\]

denote the Chebyshev constants for \(1/f\) in \([0, + \infty)\). Then, for any sequence \(\{m(n)\}_{n=0}^{\infty}\) with \(0 \leq m(n) \leq n\) for each \(n \geq 0\),

\[
\lim_{n \to \infty} (\lambda_{m(n), n})^{1/n} \leq \frac{1}{2^{1/\rho}} < 1.
\]

Moreover,

\[
\lim_{n \to \infty} (\lambda_{0, n})^{1/n} \geq \frac{1}{2^{2+1/\rho}}.
\]

The results of Theorem 2 give then sufficient conditions on \(f(z)\) so that its associated Chebyshev constants \(\lambda_{m(n), n}\) of (1.6), \(0 \leq m(n) \leq n\,\), converge geometrically to zero as \(n \to \infty\). In the spirit of Bernstein's classical inverse-type theorems for polynomial and trigonometric polynomial approximation on finite intervals, the first aim of this paper is to give necessary conditions on \(f(x)\) for this geometric convergence on the infinite interval \([0, + \infty)\). This will be discussed in §2. In addition, similar necessary conditions for geometric convergence on the infinite interval \((-\infty, + \infty)\) will also be given.

Upon examining the results of Theorem 2, we see that the bounds of (1.7)--(1.8) depend upon \(\rho\), but do not depend upon \(B\). This suggests that the results
of Theorem 2, giving sufficient conditions on the entire function \( f(z) \) to ensure geometric convergence to zero of the Chebyshev constants, can be generalized to a wider class of entire functions. The second aim of this paper is to consider just such generalizations of Theorem 2. This will be discussed in §3. Finally, in §4 quasi-analytic extensions in the sense of S. N. Bernstein will be given.

2. A converse theorem. To establish a converse result to Theorems 1–2 which gives necessary conditions of \( f \) such that the Chebyshev constants \( \lambda_{m,n} \) of \( 1/f \) converge geometrically to zero, we first introduce some notation. For given \( r > 0 \) and \( s > 1 \), let \( \mathcal{S}(r, s) \) denote the unique open ellipse in the complex plane with foci at \( x = 0 \) and \( x = r \) and semimajor and semiminor axes \( a \) and \( b \) such that \( b/a = (s^2 - 1)/(s^2 + 1) \). In more familiar notation,

\[
(2.1) \quad z = x + iy \in \mathcal{S}(r, s) \quad \text{iff} \quad \frac{(x - r/2)^2}{(r/4)(s + 1/s)^2} + \frac{y^2}{(r/4)(s - 1/s)^2} < 1.
\]

If \( F(z) \) is any entire function, we denote by \( \widetilde{M}_F(r, s) \) the maximum modulus of \( F \) in \( \mathcal{S}(r, s) \), i.e.

\[
(2.2) \quad \widetilde{M}_F(r, s) = \sup \{ |F(z)| : z \in \mathcal{S}(r, s) \}.
\]

We now state one of our main results.

**Theorem 3.** Let \( f(x) \) be a real continuous function (not \( \equiv 0 \)) on \( [0, +\infty) \), and assume that there exist a sequence of real polynomials \( \{ p_n(x) \}_{n=0}^{\infty} \) with \( p_n \in \mathbb{R}^n \) for each \( n \geq 0 \), and a real number \( q > 1 \) such that

\[
(2.3) \quad \lim_{n \to \infty} \left\{ \frac{1}{p_n(x)} - \frac{1}{f(x)} \right\}_{L^\infty[0, \infty]}^{1/n} \leq \frac{1}{q} < 1.
\]

Then, there exists an entire function \( F(z) \) with \( F(x) = f(x) \) for all \( x \geq 0 \), and \( F \) is of finite order \( \rho \), i.e.

\[
(2.4) \quad \lim_{r \to \infty} \frac{\ln \ln M_F(r)}{\ln r} \equiv \rho < \infty.
\]

In addition, for every \( s > 1 \), there exist constants \( K = K(s, q) > 0 \), \( \theta = \theta(s, q) > 1 \), and \( r_0 = r_0(s, q) > 0 \) such that

\[
(2.5) \quad \widetilde{M}_F(r, s) \leq (K \| f \|_{L^\infty[0, r]})^\theta \quad \text{for all} \quad r \geq r_0.
\]

If, for each \( s > 1 \), \( \tilde{\theta}(s) \) is defined by

\[
(2.6) \quad \lim_{r \to \infty} \left\{ \frac{\ln \tilde{M}_F(r, s)}{\ln \| f \|_{L^\infty[0, r]}} \right\} = \tilde{\theta}(s)
\]
when \( \| f \|_{L_\infty[0,r]} \) is unbounded as \( r \to \infty \), and \( \tilde{\theta}(s) = 1 \) otherwise, then the order \( \rho \) of \( F \) satisfies

\[
\rho \leq \inf_{s > 1} \left\{ \frac{\ln \tilde{\theta}(s)}{\ln \left( \frac{1/4 + 1/4(s + 1/s)}{s} \right)} \right\},
\]

and this upper bound for the order \( \rho \) is in general best possible.

Proof. For any \( q_1 \) with \( q > q_1 > 1 \), it follows from (2.3) that there exists a positive integer \( n_1(q_1) \) such that

\[
\| 1/p_n - 1/f \|_{L_\infty[0,\infty]} \leq 1/q_1^n \quad \text{for all} \quad n \geq n_1(q_1).
\]

Next, define

\[
m_j(r) = \| f \|_{L_\infty[0,r]} \quad \text{for each} \quad 0 \leq r < \infty.
\]

Fixing \( r > 0 \), the fact that \( q_1 \) exceeds unity implies that there exists a least integer \( n_2(r) \) such that

\[
q^n_1 > q^n_1 - m_2(r) \geq q^n_1 / 2 \quad \text{for all} \quad n \geq n_2(r).
\]

With \( n_3 = \max \{ n_1(q_1), n_2(r) \} \), a simple manipulation of the inequality of (2.8), coupled with the second inequality of (2.10), gives us that

\[
|p_n(x) - f(x)| \leq m_2^n(r) / (q^n_1 - m_2(r)) \leq 2m_2^n(r) / q^n_1
\]

for any \( n \geq n_3 \) and all \( x \) with \( 0 \leq x \leq r \) and \( f(x) \neq 0 \). But from the continuity of \( f \), this implies that

\[
\| p_n - f \|_{L_\infty[0,r]} \leq 2m_2^n(r) / q^n_1 \quad \text{for all} \quad n \geq n_3.
\]

In order to apply results of S. N. Bernstein, we now make the change of variables \( r(1 + t) / 2 = x, \quad 0 \leq x \leq r, \quad -1 \leq t \leq 1 \), and define

\[
b(t; r) = f(r(1 + t)/2).
\]

Because \( f \) is real and continuous on \([0, +\infty)\), \( b(t; r) \) is a real continuous function on \([-1, +1] \) for each parameter \( r \geq 0 \). If

\[
E_n(b(\cdot, r)) = \inf_{\sigma_n \in \pi_n} \| \sigma_n - b(\cdot, r) \|_{L_\infty[-1,1]}
\]

denotes the error in the best Chebyshev polynomial approximation in \( \pi_n \) to \( b(\cdot, r) \) on \([-1, +1] \), then the inequality of (2.11) immediately implies that

\[
E_n(b(\cdot, r)) \leq 2m_2^n(r) / q^n_1 \quad \text{for all} \quad n \geq n_3.
\]

But, as \( r > 0 \) is fixed and \( q_1 \) is an arbitrary number with \( q > q_1 > 1 \), this implies that
\[ \lim_{n \to \infty} \| E_n b(\cdot; r) \|^{1/n} \leq 1/q \quad \text{for each } r > 0. \]

Hence, using a result of S. N. Bernstein (cf. [3, p. 86]), this means that \( b(t; r) \) can, for each \( r > 0 \), be extended to an analytic function in the open ellipse \( E_q \) with foci \( \pm 1 \) and semimajor and semiminor axes \( a \) and \( b \) such that \( a + b = q > 1 \).

Because
\[ z = x + iy \in E_q \quad \text{iff} \quad \frac{x^2}{[\frac{1}{2}(q + 1/q)]^2} + \frac{y^2}{[\frac{1}{2}(q - 1/q)]^2} < 1, \]

this means, using (2.1) and (2.12), that \( f \) can be extended to a function \( F(z) \) which is analytic in the ellipse \( E(r, q) \). But as \( r \) is an arbitrary positive real number, and any complex number \( w \) is, as is readily verified, in \( E(r, q) \) for all \( r \) sufficiently large, then \( F(z) \) is evidently analytic in the whole complex plane, i.e. \( F \) is an entire function, which proves the first part of the theorem.

Continuing, let \( \sigma_n(t; r) = p_n(t(1 + t/2)]. \) Then, the inequality of (2.11) can be rewritten as
\[ \| \sigma_n(\cdot; r) - b(\cdot; r) \|_{L_\infty[-1,+1]} \leq 2m_2(r)/q_1^n \quad \text{for all } n \geq n_3. \]

With the triangle inequality, and the definitions of (2.9) and (2.12), we deduce from the above inequality that
\[ \| \sigma_n(\cdot; r) \|_{L_\infty[-1,+1]} \leq m_1(r) + 2m_2(r)/q_1^n \quad \text{for all } n \geq n_3, \]

and
\[ \| \sigma_{n+1}(\cdot; r) - \sigma_n(\cdot; r) \|_{L_\infty[-1,+1]} \leq 2(1 + q_1)m_2(r)/q_1^{n+1} \quad \text{for all } n \geq n_3. \]

Making use of another result of S. N. Bernstein (cf. [3, p. 92]), the above inequalities can, for any \( s \) with \( 1 < s < q_1 \), be extended to any \( z \in E_s \) by
\[ (2.13) \quad |\sigma_n(z; r)| \leq (m_1(r) + 2m_2(r)/q_1^n) s^n \quad \text{for all } n \geq n_3, \quad z \in E_s, \]

and
\[ (2.14) \quad |\sigma_{n+1}(z; r) - \sigma_n(z; r)| \leq 2(1 + q_1)m_2(r)(s/q_1)^{n+1} \quad \text{for all } n \geq n_3, \quad z \in E_s. \]

From these inequalities, the series
\[ \sigma_{n_3}(z; r) + \sum_{n = n_3}^{\infty} |\sigma_{n+1}(z; r) - \sigma_n(z; r)| \]

necessarily converges uniformly on \( E_s \) to an analytic function which, as is easily seen, is \( b(z; r). \) With the bounds of (2.13) and (2.14) applied to the above sum, it follows that
\[ |b(z; r)| \leq \left( \frac{m_f(r)}{q_1} + \frac{2m_f^2(r)}{q_1^3} \right) s^{n_3} + 2(1 + q_1) m_f^2(r) \left( \frac{s}{q_1 - s} \right)^{n_3+1} \]

for all \( z \in \mathbb{D}_s \). Noting from (2.10) that \( 2m_f(r)/q_1^{-n_3} \leq 1 \), the above inequality becomes

\[ |b(z; r)| \leq m_f(r) \left\{ 2 + (1 + q_1) s/(q_1 - s) \right\} s^{n_3} \quad \text{for all } z \in \mathbb{D}_s, \]

However, from the definitions of (2.1) and (2.12), this means that

\[ |F(z)| \leq m_f(r) \left\{ 2 + (1 + q_1) s/(q_1 - s) \right\} s^{n_3} \quad \text{for all } z \in \mathbb{D}(r, s), \]

which in turn, from (2.2), implies that

\[ \tilde{M}_P(r, s) \leq m_f(r) \left\{ 2 + (1 + q_1) s/(q_1 - s) \right\} s^{n_3} \quad \text{for all } r > 0, \text{ all } 1 < s < q_1. \]

Consider now the nondecreasing function of \( r, m_f(r) \). We first assume that \( m_f(r) \) is unbounded as \( r \to \infty \). Then, for any \( r_0 \) sufficiently large with \( m_f(r_0) > 1 \), define

\[ r = r(r_0) = \left( \frac{\ln s}{\ln q_1} \right) \left\{ \frac{-\ln 2}{\ln m_f(r_0)} + 1 \right\} \]

It can be verified, using the nondecreasing nature of \( m_f(r) \) and the fact from (2.10) that \( n_3 = m_2(r) \leq (2q_1 + \ln m_f(r))/\ln q_1 \), that \( s^{n_3} \leq (m_f(r))^{1+r} \) for all \( r \geq r_0 \). In other words, from (2.15),

\[ \tilde{M}_P(r, s) \leq \left\{ 2 + (1 + q_1) s/(q_1 - s) \right\} [(m_f(r))]^{1+r} \quad \text{for all } r \geq r_0, \text{ all } 1 < s < q_1. \]

Because \( m_f(r) \), by assumption, is nondecreasing and unbounded as \( r \to \infty \), it is clear from (2.16) that, for any \( s \) chosen with \( 1 < s < q_1 \), there is an \( r_0 > 0 \) such that \( r(r_0) \leq 1 \), and hence with \( K = K(q_1, s) = \left\{ 2 + (1 + q_1) s/(q_1 - s) \right\}^{1/2} \), we have that

\[ \tilde{M}_P(r, s) \leq \{ Km_f(r) \}^{2} \quad \text{for all } r \geq r_0, \text{ all } 1 < s < q_1. \]

For the remaining case when \( m_f(r) \) is bounded, i.e. \( m_f(r) \leq \sigma < \infty \) for all \( r \geq 0 \), we see from (2.10) that \( n_3 \), and hence \( n_3 \), is independent of \( r \). In this case, (2.15) becomes

\[ \tilde{M}_P(r, s) \leq \sigma K^2 s^{n_3} \quad \text{for all } r > 0, \text{ all } 1 < s < q_1. \]

As is readily verified, if for \( s > 1 \), \( \mu = \mu(s) \) is defined by

\[ \mu = \frac{1}{2} \left\{ 1 + \frac{1}{2}(s + 1/s) \right\} > 1, \]

then the open disk \( \{ z : |z| < (\mu - 1)r \} \) is contained in \( \mathbb{D}(r, s) \). Consequently, by the maximum modulus principle,
Combining the inequalities of (2.18) and (2.20), we see that \( M_F(r) \) is bounded for all \( r > 0 \), so that \( F(z) = c \) for all \( z \). Because \( F(x) \) agrees with \( f(x) \) on \([0, +\infty)\), the assumption that \( f(x) \) is not identically zero on \([0, +\infty)\) gives us that \( F(z) = c \neq 0 \) for all \( z \), and in this case, the inequality of (2.17) is trivially satisfied.

We now show that the entire function \( F \) is of finite order \( \rho \), where \( \rho \) is defined by (2.4). It is geometrically evident from the definition of \( \mu \) in (2.19) that the ellipse \( E(r/\mu, s) \), for \( 1 < s < q \), contains all points of the interval \((0, r)\) for any \( r > 0 \). Hence, by the maximum modulus principle again, \( m_f(r) \leq \tilde{M}_F(r/\mu, s) \), and thus from (2.17), \( \tilde{M}_F(r, s) \leq (KM_F(r/\mu, s))^2 \) for all \( r \geq \mu r_0 \). From induction on the above inequality and the inequality of (2.20), it follows that

\[
M_F((\mu - 1) r) \leq \tilde{M}_F(r, s) \leq (\tilde{M}_F(r/\mu, s))^2m \quad \text{for all } r \geq \mu^m r_0.
\]

In a similar way, we have that

\[
m_f(r) \leq (\tilde{M}_F(r/\mu, s))^2m \quad \text{for all } r \geq \mu^m r_0.
\]

For each \( r \geq r_0 \), choose the nonnegative integer \( m \) such that \( \mu^{m+1} > r/r_0 \geq \mu^m \).

A short calculation based on the inequalities of (2.21) gives us that

\[
\rho = \lim_{r \to \infty} \frac{\ln \ln M_F(r)}{\ln r} \leq \frac{\ln 2}{\ln \mu},
\]

proving that \( F(x) \) is an entire function of finite order \( \rho \leq (\ln 2)/(\ln \mu) \). But as this inequality is valid for all \( s \) with \( 1 < s < q_1 < q \), it holds also for \( s = q \), i.e. from (2.19),

\[
\rho \leq \frac{\ln 2}{\ln [\frac{1}{2}(1 + \frac{1}{2}(q + 1/q))]}.
\]

Continuing the proof of Theorem 3, we now establish the inequality of (2.5). We already know from (2.17) that for any \( s \) with \( 1 < s < q_1 \), \( \tilde{M}_F(r, s) \leq (Km_f(r))^2 \) for all \( r \geq r_0 \). Next, it is easy to verify geometrically that \( E(r, s) \subset E(r', s') \) for \( r' \geq r \), where \( s' \) and \( s \) are connected by (cf. (2.19))

\[
(\mu (s') - 1) r' = (\mu (s) - 1) r.
\]

Given any \( s > 1 \), choose \( \alpha = r'/r \geq 1 \) such that \( s' \), determined from (2.25), satisfies \( 1 < s' < q_1 \); this is always possible for \( \alpha \) sufficiently large. Then, fixing \( \alpha \) and \( s' \), it follows from \( E(r, s) \subset E(r', s') = E(\alpha r, s') \) that \( \tilde{M}_F(r, s) \leq \tilde{M}_F(\alpha r, s') \). But as \( 1 < s' < q_1 \), we can apply (2.17), i.e. \( m_f(r) \leq \tilde{M}_F(r, s) \leq M_F(\alpha r, s') \leq (Km_f(\alpha r))^2 \) for all \( r \geq r_0 / \alpha \), the first inequality following from the fact that \([0, r] \subset E(r, s) \). Making use of the inequality of (2.22), we deduce from the above inequality that
\[ \hat{M}_F(r, s) \leq |K^2 m_m(\alpha r/\mu^m(s'))|^{2m+1} \text{ for all } \alpha r \geq \mu^m(s) r_0. \]

Now, choose \( m \) sufficiently large so that \( \alpha \leq \mu^m(s') \). For this choice of \( m \), we have \( \hat{M}_F(r, s) \leq |K^2 m_m(r)|^{2m+1} \) for all \( r \geq r_0 \), the desired result of (2.5).

To establish the inequality of (2.7), it is sufficient to assume that \( \| f \|_{L_{\infty}[0, r]} \) is unbounded as \( r \to \infty \). In this case, if follows from (2.5) that

\[ \hat{\theta}(s) \equiv \lim_{r \to \infty} \left\{ \frac{\ln \hat{M}_F(r, s)}{\ln \| f \|_{L_{\infty}[0, r]}} \right\} \geq 1 \]

is finite for each \( s > 1 \). For \( s > 1 \) fixed, if follows from (2.26) that for any \( \epsilon > 0 \), there is an \( r_0(\epsilon, s) > 0 \) such that

\[ \hat{M}_F(r, s) \leq \| f \|_{L_{\infty}[0, r]}^{(\hat{\theta}(s) + \epsilon)} \text{ for all } r \geq r_0(\epsilon, s), \]

from which it follows, as in (2.21), that

\[ \hat{M}_F(r, s) \leq \| M_F(r/\mu^m(s), s) \|_{L_{\infty}[0, r]}^{(\hat{\theta}(s) + \epsilon)m} \text{ for all } r \geq \mu^m(s) \cdot r_0(\epsilon, s). \]

The above inequality, as in the proof of (2.23), gives us that

\[ \rho = \lim_{r \to \infty} \frac{\ln \hat{M}_F(r)}{\ln r} \leq \frac{\ln \hat{\theta}(s)}{\ln \mu(s)} \text{ for each } s > 1, \]

from which we conclude that

\[ \rho \leq \inf_{s > 1} \left\{ \frac{\ln \hat{\theta}(s)}{\ln \mu(s)} \right\}, \]

the desired result of (2.7).

Finally, to show that the inequality of (2.7) is in general best possible, consider any entire function \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) of perfectly regular growth \( \langle \rho, B \rangle \) (cf. (1.5)) with \( a_0 > 0, a_k \geq 0 \) for all \( k \geq 0 \). By Theorem 2, the assumptions of Theorem 3 are fulfilled. Because \( f \) is of perfectly regular growth \( \langle \rho, B \rangle \), it follows from (1.5) that for any \( \delta > 0 \),

\[ \lim_{r \to \infty} \frac{\ln M_f(\delta r)}{\ln M_f(r)} = \delta^\rho. \]

But, because the coefficients \( a_k \) of \( f \) are all nonnegative, then \( \hat{M}_f(r, s) = M_f(\mu(s) \cdot r) \) and \( \| f \|_{L_{\infty}[0, r]} = M_f(r) \) for all \( r > 0, \) all \( s > 1 \). Thus, from (2.27) and the definition of \( \hat{\theta}(s) \) in (2.6), it follows that \( \hat{\theta}(s) = (\mu(s))^\rho \), so that

\[ \rho = \left\{ \frac{\ln \hat{\theta}(s)}{\ln \mu(s)} \right\} \text{ for each } s > 1, \]

i.e. equality holds in (2.7). Q.E.D.

The following are consequences of Theorem 3 and its proof. The first is merely a sharpened restatement of the inequality of (2.24).
Corollary 1. If \( f(x) \) satisfies the hypotheses of Theorem 3, and \( \rho > 0 \) is the order of the entire function \( F(z) \), then
\[
q \leq 2^{1/\rho + 1} - 1 + \left( (2^{1/\rho + 1} - 1)^2 - 1 \right)^{1/2},
\]
with strict inequality holding if \( F \) is of type \( B = +\infty \) (cf. (3.8)).

We remark that the bound of (2.28) simultaneously improves the upper bound \( q \leq 6 \) of (1.4) of Theorem 1 for the case \( \rho = 1 \), as well as the upper bound \( q \leq 2^{n+1}/\rho \) of (1.8) of Theorem 2 for the more general case \( \rho > 0 \).

Corollary 2. If \( f(x) \) satisfies the hypotheses of Theorem 3 and \( \|f\|_{L_\infty[0,\infty]} \) is bounded, then \( f(x) = c \neq 0 \).

The point of Corollary 2 is that it is easy to give examples of entire functions which are real on \([0, +\infty)\) which cannot possess the property of (2.3). For example \( F(z) = \cos z + 2 \) is such a function. A less obvious example which makes use of the inequality of (2.5) is \( F(z) = (z + 1)(2 + \cos z) \), which is an entire function of order \( \rho = 1 \), for which \( r + 1 \leq \pi(r) \leq 3(r + 1) \) for all \( r \geq 0 \). But since \( |F(z)| \) grows exponentially along the imaginary axis, it is clear that the necessary condition of (2.5) of Theorem 3 cannot hold for all \( r \) sufficiently large.

It is of interest to know that there is an analogue of Theorem 3 which deals with approximations on the infinite interval \((-\infty, +\infty)\). In fact, this extension to \((-\infty, +\infty)\) directly follows the steps of the proof of Theorem 3, with the interval \([0, r]\) being replaced by \([-r, +r]\). If \( \mathcal{E}(r, s) \), for \( r > 0 \) and \( s > 1 \), denotes the open ellipse in the complex plane with foci at \( \pm r \) and semimajor and semiminor axes \( a \) and \( b \) such that \( b/a = (s^2 - 1)/(s^2 + 1) \), then, in analogy with (2.2), define
\[
\hat{\mathcal{E}}(r, s) = \sup \{|F(z)| : z \in \mathcal{E}(r, s)|.
\]

We then have

Theorem 4. Let \( f(x) \) be a real continuous function (not \( \equiv 0 \)) on \((-\infty, +\infty)\), and assume that there exist a sequence of real polynomials \( \{p_n(x)\}_{n=0}^{\infty} \), with \( p_n \in \Pi_n \) for each \( n \geq 0 \), and a real number \( q > 1 \) such that
\[
\lim_{n \to \infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L_\infty[-\infty, +\infty]} \right\}^{1/n} = \frac{1}{q} < 1.
\]

Then, there exists an entire function \( F(x) \) with \( F(x) = f(x) \) for all \( x \) with \(-\infty < x < +\infty\), and \( F \) is of finite order \( \rho \) (cf. (2.24)). In addition, for every \( s > 1 \), there exist constants \( K = K(s, q) > 0 \), \( \theta = \theta(s, q) > 1 \), and \( r_0 = r_0(s, q) > 0 \) such that
(2.31) \[
\tilde{M}_f(r, s) \leq \left( K \| f \|_{L_\infty([-r, r])} \right)^\theta \quad \text{for all } r \geq r_0.
\]

If, for each \( s > 1 \), \( \tilde{\theta}(s) \) is defined by

(2.32) \[
\lim_{r \to \infty} \left\{ \frac{\ln \tilde{M}_f(r, s)}{\ln \| f \|_{L_\infty([-r, r])}} \right\} = \tilde{\theta}(s)
\]

when \( \| f \|_{L_\infty([-r, r])} \) is unbounded as \( r \to \infty \), and \( \tilde{\theta}(s) \equiv 1 \) otherwise, then the order \( \rho \) of \( F \) satisfies

(2.33) \[
\rho \leq \inf_{s > 1} \left\{ \frac{\ln \tilde{\theta}(s)}{\ln \left[ s/2 + 1/2s \right]} \right\},
\]

and this upper bound for the order \( \rho \) is in general best possible.

We conclude this section with the remark that \( f(z) = e^{z^2} = \sum_{k=0}^{\infty} a_k z^k \), of perfectly regular growth (\( \rho = 2, B = 1 \)) and \( a_k \geq 0 \) for all \( k \geq 0 \), necessarily satisfies (by virtue of Theorem 2) the hypotheses of Theorem 4, and gives the case of equality in (2.32).

3. Some sufficient conditions for geometric convergence. In this section, we make use of the property of (2.5) of Theorem 3 to establish new sufficient conditions on \( f \) for the geometric convergence to zero of its Chebyshev constants, as in (2.3).

**Theorem 5.** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be an entire function with \( a_0 > 0 \) and \( a_k \geq 0 \) for all \( k \geq 1 \). If there exist real numbers \( A > 0, s > 1, \theta > 0, \) and \( r_0 > 0 \) such that

(3.1) \[
\tilde{M}_f(r, s) \leq A (\| f \|_{L_\infty([0, r])})^\theta \quad \text{for all } r \geq r_0,
\]

then there exists a sequence of real polynomials \( \{ p_n(x) \}_{n=0}^{\infty} \) with \( p_n \in \pi_n \) for each \( n \geq 0 \), and a real number \( q \geq s^{1/(1+\theta)} > 1 \) such that

(3.2) \[
\lim_{n \to \infty} \left\{ \left( \frac{1}{|p_n(x)|} - 1 \right)^{1/n} \| f \|_{L_\infty([0, \infty])} \right\}^{1/q} = \frac{1}{q} < 1.
\]

Consequently, if \( \lambda_{m,n} \) denote the Chebyshev constants for \( 1/f \) in \([0, +\infty)\) (cf. (1.6)), then for any sequence of positive integers \( \{ m(n) \}_{n=0}^{\infty} \) with \( 0 \leq m(n) \leq n \), then

(3.3) \[
\lim_{n \to \infty} 1_{\lambda_{m(n), n}}^{1/n} \leq \frac{1}{q} < 1.
\]

**Proof.** For each \( r > 0 \), let \( q_r(x; r) \in \pi_n \) denote the best Chebyshev approximation to \( f \) in \([0, r] \), i.e.
(3.4) \[ \|f - q_n(x; r)\|_{L_{\infty}[0, r]} = \inf_{\sigma_n \in \pi_n} \|f - \sigma_n\|_{L_{\infty}[0, r]} = \delta_n(r). \]

It is well known that \( q_n(x; r) \) can be regarded as a polynomial interpolation of \( f(x) \) on \([0, r]\), i.e., there exist \( 0 < x_1(r) < x_2(r) < \cdots < x_{n+1}(r) < r \) such that \( q_n(x_j(r); r) = f(x_j(r)), \) \( 1 \leq j \leq n + 1 \). Express \( q_n(x; r) \) in Newton interpolation series form, i.e.,

\[
q_n(x; r) = f(x_1(r)) + f[x_1(r), x_2(r)](x - x_1(r)) + \cdots + f[x_1(r), \ldots, x_{j+1}(r)] \prod_{j=1}^{n} (x - x_j(r)),
\]

where \( f[x_1(r), \ldots, x_{j+1}(r)] \) are divided differences of \( f \) in the points \( x_1(r), \ldots, x_{j+1}(r) \). As is well known, \( f[x_1(r), \ldots, x_{j+1}(r)] = f^{(j)}(\xi)/j! \) where \( x_1(r) < \xi < x_{j+1}(r) \), and because of the assumption that the Taylor coefficients are all nonnegative, these divided differences are evidently all nonnegative. Thus, \( q_n(x; r) \) is monotonically increasing as a function of \( x \) for all \( x \geq r \). Next, let \( p_n(x; r) = q_n(x; r) + \delta_n(r) \) for each \( n \geq 0 \). From (3.4), it is clear that

(3.5) \[ p_n(x; r) \geq f(x) \geq f(0) > 0 \quad \text{for all} \quad x \in [0, r]. \]

Similarly, from the monotonic nature of \( p_n(x; r) \) as a function of \( x \) for all \( x \geq r \), we also have that \( p_n(x; r) \geq f(r) \) for all \( x \geq r \). But as \( f(x) \), by hypothesis, is nondecreasing on \([0, +\infty)\), we also have that \( f(x) \geq f(r) > 0 \) for all \( x \geq r \). Hence, from the above two inequalities,

(3.6) \[ \left| 1/f(x) - 1/p_n(x; r) \right| \leq 2/f(r) \quad \text{for all} \quad x \geq r. \]

For the interval \([0, r]\) on the other hand, it follows from (3.5) that

\[
\left| \frac{1}{f(x)} - \frac{1}{p_n(x; r)} \right| = \left| \frac{p_n(x; r) - f(x)}{f(x) \cdot p_n(x; r)} \right| \leq \frac{2\delta_n(r)}{f^2(0)}, \quad x \in [0, r].
\]

Next, since \( f \) is by hypothesis an entire function, another result of S. N. Bernstein (cf. [3, p. 91]) gives us that

\[ \delta_n(r) = \|f - q_n(x; r)\|_{L_{\infty}[0, r]} \leq M_j(r, s)/(s - 1)s^n \]

for any \( s > 1 \) and any \( n \geq 0 \). Now choose the \( s > 1 \) for which (3.1) is valid. Then, using the hypothesis of (3.1) and noting that \( \|f\|_{L_{\infty}[0, r]} = f(r) \), it follows from the last two inequalities that

(3.7) \[ \|1/f(x) - 1/p_n(x; r)\|_{L_{\infty}[0, r]} \leq Bf^6(r)/s^n \quad \text{for all} \quad n \geq 0, \quad r \geq r_0, \]

where \( B \equiv 2A/(s - 1)s^2(0)! \). Now, the assumption that \( f(r) \) is unbounded as \( r \to \infty \) gives us that there exist a positive integer \( n_4 \) and an \( r(n) \geq r_0 \) such that
Consequently, if we set $p_n(x) = p_k(x; r(n))$, we see that 
\[ \left\| 1/f(x) - 1/p_n(x) \right\|_{L_\infty[0,\infty]} \leq C/s^{n/(1+\theta)} \]
for all $n \geq n_\delta$, where $C = \max\{2, B\}$, from which it follows that
\[ \lim_{n \to \infty} \left\{ \left\| 1/f(x) - 1/p_n(x) \right\|_{L_\infty[0,\infty]} \right\}^{1/n} \leq \frac{1}{s^{1/(1+\theta)}} < 1, \]
the desired result of (3.2). Finally, from (1.6), it is obvious that the Chebyshev constants for $1/f$ satisfy $0 \leq \lambda_{n,n} \leq \lambda_{n-1,n} \leq \cdots \leq \lambda_{0,n}$, and the result of (3.3) follows immediately from (3.2). Q.E.D.

We remark that the assumption in Theorem 4 that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ have only nonnegative Taylor coefficients $a_k$ is probably stronger than is needed in general. For example, for $f(z) = e^z + 2e^{-z}$, which has every other Taylor coefficient negative, the conclusion of (3.2) can be shown to be valid. On the other hand, if we consider $f(z) = e^{2z} \sin z$, then $f(z)$ does satisfy the growth condition of (3.1), and has Taylor coefficients $a_k$ which are not all nonnegative. But as $f(zk) = 0$ for all nonnegative integers $k$, it is clear that (3.2) could not hold for any sequence $\{p_n(x)\}_{n=0}^{\infty}$ with $p_n \in \pi_n$ for each $n \geq 0$.

To show that the result of Theorem 5 contains the results of Theorems 1 and 2 as special cases, we prove

**Theorem 6.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with $a_0 > 0$ and $a_k \geq 0$ for all $k \geq 0$, and assume that there exist finite numbers $\rho > 0$ and $0 < b \leq B$ such that
\[ \lim_{r \to \infty} \frac{\ln M_f(r)}{r^\rho} = B, \quad \lim_{r \to \infty} \frac{\ln M_f(r)}{r^\rho} = b. \]

Then, there exist real numbers $A > 0$, $s > 1$, and $r_0 > 0$ such that
\[ \tilde{M}_f(r, s) \leq A\left\| f \right\|_{L_\infty[0,1]}^s \]
for all $r \geq r_0$. Thus, the conclusions of Theorem 5 are valid.

**Proof.** For any $0 < \epsilon < b$, choose any $0 > 1$ such that $\theta(b - \epsilon) / (B + \epsilon) > 1$. It follows from (3.8) that there exists an $r_0(\epsilon)$ such that
\[ e^{(b-\epsilon)\rho} < M_f(r) < e^{(B+\epsilon)\rho} \quad \text{for all } r \geq r_0(\epsilon). \]

Because the Taylor coefficients $a_k$ of $f$ are all nonnegative, we see that for any $s > 1$, and any $r > 0$, $\tilde{M}_f(r, s) = \tilde{M}_f(\mu r) = \tilde{M}_f(\mu)$, where $\mu > 1$ is given by (2.19). Thus, to prove the inequality of (3.9) it suffices to show that there is a $\mu > 1$ for which $M_f(\mu) \leq (M_f(\mu))^\rho$ for all $r \geq r_0(\epsilon)$. But from the inequalities of (3.10), this is true for $\mu = \left[ \theta(b - \epsilon) / (B + \epsilon) \right]^{1/\rho} > 1$. Q.E.D.

We remark that as the assumption for perfectly regular growth of $f$ in (1.5),
$0 < t$ for all $t > 0$, then as before, $f(t)$ is bounded. But now show that if $\mathcal{L}$ is applied, we have shown that Theorem 5 is applicable. Because $\mathcal{L}$ is applied, we have shown that Theorem 6 cannot be applicable. Thus, there is no $\epsilon$ such that $\mathcal{L}$ cannot be applicable. Hence, we have shown that

$\int_0^\infty \frac{1}{1+\|u\|^2} \, dt = \infty$.

As is readily verified, it follows that

$\int_0^\infty \frac{1}{1+\|u\|^2} \, dt = \infty$.

Finally, consider the unique function defined by

$\phi(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$

Then, for all $x > 0$ such that $W > (s + 1) \| \Phi \|$, it follows that $W > (s + 1) \| \Phi \|$, which is equivalent to $x > 0$. Thus, from Theorem 6, we have that

$\int_0^\infty \frac{1}{1+\|u\|^2} \, dt = \infty$.

Again, in the proof of Theorem 6, the non-negative Taylor coefficients of

$\phi(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$

follows from (3.12) that there is an $\epsilon$ such that $x > 0$, for any $\epsilon > 0$. Choose such an $\epsilon$ that $x > 0$.

Thus, the Chebyshev constants for $x$ satisfy

$0 = \left\{ \begin{array}{ll}
\| \frac{\partial^m}{\partial t^m} \phi \|_{L^2} & \text{if } m \geq 0, \\
\| \frac{\partial^m}{\partial t^m} \phi \|_{L^2} & \text{if } m < 0.
\end{array} \right.$

Then, there exists a sequence of real polynomials

$Q = \frac{\partial^m}{\partial t^m} \phi$

such that

$\| Q \|_{L^2} \leq \| \frac{\partial^m}{\partial t^m} \phi \|_{L^2} = \infty$.

and

$\lim_{n \to \infty} \frac{\partial^m}{\partial t^m} \phi(t) = \infty$.

Theorem 7. Let $\Phi$ be an entire function of order $p$ with $0 = p < \infty$ and $\Phi(z) = \infty$ for all $z \in \mathbb{C}$.

Then, Theorem 7 is applicable and the conditions of Theorem 5, and hence Theorem 6, are satisfied.
Thus, as in the proof of Theorem 3 (cf. 2.11), we have that
\[ (i) \frac{1}{q} t < \gamma < r \quad \text{for all} \quad z \quad \frac{1}{q} t < \gamma < \frac{1}{q} \]

where \( \gamma \) is the least integer such that
\[ \frac{1}{q} t < \gamma < \frac{1}{q} \quad \text{for all} \quad z \quad \frac{1}{q} t < \gamma < \frac{1}{q} \]

sufficiently large \( \gamma \) with \( \frac{1}{q} t < \gamma < \frac{1}{q} \).

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Proo. For any \( q \), \( t \), \( a > 0 \), \( t < a \) such that \( \frac{1}{q} t < \gamma < \frac{1}{q} \)
there exists a sequence of real numbers \( \{ \gamma_i \} \)
for which \( \frac{1}{q} t < \gamma_i < \frac{1}{q} \).

Then, \( \limsup \{ \gamma_i \} = \gamma \), \( \liminf \{ \gamma_i \} = \gamma \), \( \gamma \) finite.

Thus, as in the proof of Theorem 3 (cf. 2.11), we have that

\[ (i) \frac{1}{q} t < \gamma < r \quad \text{for all} \quad z \quad \frac{1}{q} t < \gamma < \frac{1}{q} \]

where \( \gamma \) is the least integer such that
\[ \frac{1}{q} t < \gamma < \frac{1}{q} \quad \text{for all} \quad z \quad \frac{1}{q} t < \gamma < \frac{1}{q} \]

sufficiently large \( \gamma \) with \( \frac{1}{q} t < \gamma < \frac{1}{q} \).

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Theorem 3, Let be a real continuous function on \( [0, \infty) \) with a finite number of points \( x_1 < x_2 < \cdots < x_n \). In the case of \( \frac{1}{q} t < \gamma < \frac{1}{q} \),\( \gamma \) finite.

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Theorem 5 is, however, applicable.

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Theorem 6, The hypotheses of Theorem 6, but not in the previous example.

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Theorem 7. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 8. Here, II = \( \delta \) and II = \( \delta \) apply.

---

Theorem 9. Here, II = \( \delta \) and II = \( \delta \) apply.

---

Theorem 10. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 11. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 12. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 13. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 14. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 15. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 16. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 17. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 18. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 19. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 20. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 21. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 22. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 23. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 24. Here, II = \( \delta \) and II = \( \delta \) apply.

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Theorem 25. Here, II = \( \delta \) and II = \( \delta \) apply.
REFERENCES

\[ \lim_{u \to \infty} \left( \frac{\alpha(u) \rho(u)}{u} \right) = 0 \]

\[ \lim_{u \to \infty} \frac{a(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{b(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{c(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{d(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{e(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{f(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{g(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{h(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{i(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{j(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{k(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{l(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{m(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{n(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{o(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{p(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{q(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{r(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{s(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{t(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{u(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{v(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{w(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{x(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{y(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{z(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{A(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{B(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{C(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{D(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{E(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{F(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{G(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{H(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{I(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{J(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{K(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{L(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{M(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{N(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{O(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{P(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{Q(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{R(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{S(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{T(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{U(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{V(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{W(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{X(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{Y(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{Z(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{A'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{B'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{C'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{D'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{E'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{F'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{G'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{H'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{I'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{J'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{K'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{L'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{M'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{N'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{O'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{P'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{Q'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{R'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{S'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{T'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{U'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{V'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{W'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{X'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{Y'(u)}{u} = 0 \]

\[ \lim_{u \to \infty} \frac{Z'(u)}{u} = 0 \]