Minimal G-Functions. II

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ABSTRACT

The concept of a G-function, introduced by Nowosad and Hoffman, gives an appropriate setting for many generalizations of the Gershgorin Circle Theorem. In this paper, we extend our previous results for minimal G-functions to the partitioned matrix case.

1. G-Functions

For n a fixed positive integer, n ≥ 2, let \( \mathbb{C}^n \) denote the n-dimensional vector space of all column vectors \( x = (x_1, x_2, \ldots, x_n)^T \), and let \( \mathbb{C}^{n,n} \) denote the set of all \( n \times n \) complex matrices. Let \( \mathscr{P}_n \) be the collection of all functions \( f = (f_1, f_2, \ldots, f_n) \) such that for each \( i, i = 1, 2, \ldots, n, f_i: \mathbb{C}^{n,n} \to \mathbb{R}_+ \), i.e. \( 0 ≤ f_i(A) < \infty \) for each \( A \in \mathbb{C}^{n,n} \), and such that \( f_i \) depends only on the moduli of the off-diagonal entries of the matrices, i.e. if \( B = (b_{i,j}) \in \mathbb{C}^{n,n} \) and \( A = (a_{i,j}) \in \mathbb{C}^{n,n} \) satisfy \( |b_{i,j}| = |a_{i,j}| \) for all \( i, j = 1, 2, \ldots, n, i \neq j \), then \( f_i(B) = f_i(A), i = 1, 2, \ldots, n \).

As in Nowosad [8], Hoffman [4], and Carlson and Varga [1], we say that \( f \in \mathscr{P}_n \) is a G-function if, for any \( A = (a_{i,j}) \in \mathbb{C}^{n,n} \) satisfying

\[
|a_{i,j}| > f_i(A), \quad i = 1, 2, \ldots, n,
\]

\( A \) is nonsingular. Let \( \mathscr{G}_n \) denote the set of all G-functions \( f \in \mathscr{P}_n \). For example, if \( x \in \mathbb{C}^n \) has positive components, written \( x > 0 \), then \( r^x = (r_1^x, r_2^x, \ldots, r_n^x) \) and \( c^x = (c_1^x, c_2^x, \ldots, c_n^x) \) are well-known G-functions.*

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where

\[ r_i^x(A) = \frac{1}{x_i} \sum_{j=1 \atop j \neq i}^n |a_{i,j}|x_j, \quad c_i^x(A) = \frac{1}{x_i} \sum_{j=1 \atop j \neq i}^n |a_{j,i}|x_j, \quad i = 1, 2, \ldots, n, \]

for any \( A = (a_{i,j}) \in \mathbb{C}^{n \times n}. \)

If \( f \in \mathcal{P}_n \) and if \( A = (a_{i,j}) \in \mathbb{C}^{n \times n} \), then \( \mathcal{M}^f(A) = (\alpha_{i,j}) \in \mathbb{C}^{n \times n} \) is defined as the matrix whose entries are

\[ \alpha_{i,j} = - |a_{i,j}| \quad \text{for all} \quad i \neq j, \quad \alpha_{i,i} = f_i(A), \quad i, j = 1, 2, \ldots, n. \]

An important tool in Carlson and Varga [1] is the following result, which shows the close relationship of \( G \)-functions and \( M \)-matrices.\(^*\)

**Proposition 1.** Let \( f \in \mathcal{P}_n \). Then \( f \in \mathcal{G}_n \) if and only if \( \mathcal{M}^f(A) \) is a (possibly singular) \( M \)-matrix for every \( A \in \mathbb{C}^{n \times n} \).

In the next section, we define the notion of a \( G \)-function for partitioned matrices, and prove a result analogous to Proposition 1. This will then be used to show that other results of [1] can be extended to the partitioned matrix case.

### 2. \( G \)-FUNCTIONS FOR PARTITIONED MATRICES

Let \( N \) be a fixed positive integer, \( N \geq n \). Let \( \pi \) denote

(i) a set \( P_1, P_2, \ldots, P_n \) of nonzero projections on \( \mathbb{C}^N \) for which \( P_iP_j = 0 \) for all \( i, j = 1, 2, \ldots, n, i \neq j \), and \( P_1 + P_2 + \cdots + P_n = I \), the identity operator on \( \mathbb{C}^N \), and

(ii) a set \( \psi_1, \psi_2, \ldots, \psi_n \) of vector norms on, respectively, the subspaces \( W_i \equiv P_i \mathbb{C}^N \) of \( \mathbb{C}^N, i = 1, 2, \ldots, n. \)

In the sequel, we may assume without loss of generality that

\[ W_i = \text{span}\{e_{k_{i-1}+1}, e_{k_{i-1}+2}, \ldots, e_{k_i}\}, \quad i = 1, 2, \ldots, n, \]

where \( e_j = (\delta_{j,1}, \delta_{j,2}, \ldots, \delta_{j,N})^T \), and where \( k_0 = 0 < k_1 < \cdots < k_n = N. \)

\(^*\) \( A = (a_{i,j}) \in \mathbb{C}^{n \times n} \) is a (possibly singular) \( M \)-matrix if and only if \( A \) is real with \( a_{i,j} \leq 0 \) for all \( i, j = 1, 2, \ldots, n, i \neq j, \) and \( A + \text{diag}(d_1, d_2, \ldots, d_n) \) is nonsingular whenever \( d_i > 0, i = 1, 2, \ldots, n. \) If \( A \) is an irreducible \( M \)-matrix, then \( A + \text{diag}(d_1, d_2, \ldots, d_n) \) is nonsingular whenever \( d_i \geq 0, i = 1, 2, \ldots, n, \) and \( d_1 + d_2 + \cdots + d_n > 0. \)
Any \( B \in \mathbb{C}^{N,N} \) has a block-matrix representation

\[
B = \begin{bmatrix}
B_{1,1} & \cdots & B_{1,n} \\
\vdots & & \vdots \\
B_{n,1} & \cdots & B_{n,n}
\end{bmatrix},
\]

where, for \( i, j = 1, 2, \ldots, n \), \( B_{i,j} : W_j \to W_i \) may be thought of as \( P_i B P_j \), with domain restricted to \( W_j \) and range considered as a subspace of \( W_i \).

Given \( B_{i,j} : W_j \to W_i \), we define, for \( i, j = 1, 2, \ldots, n \),

\[
M(B_{i,j}) = \sup\{\psi_j(B_{i,j}x) : x \in W_j \text{ with } \psi_j(x) \leq 1\},
\]

\[
m(B_{i,j}) = \inf\{\psi_j(B_{i,j}x) : x \in W_j \text{ with } \psi_j(x) \geq 1\}.
\]

When \( i = j \), the numbers \( m(B_{i,i}) \) are the so-called "reciprocal norms" (cf. Friedler and Pták [3]) in that if \( B_{i,i} \) is nonsingular, then \( m(B_{i,i}) = [M(B_{i,i})]^{-1} \) (and if \( B_{i,i} \) is singular, \( m(B_{i,i}) = 0 \)).

Let \( \mathbb{C}_x^{N,N} \) denote the subset of \( \mathbb{C}^{N,N} \) of matrices \( B = (B_{i,j}) \) for which each \( B_{i,i} \) is nonsingular, \( i = 1, 2, \ldots, n \). Let \( \mathcal{F}_x \) be the collection of all functions \( F = (F_1, F_2, \ldots, F_n) \) for which

(i) \( F_i : \mathbb{C}_x^{N,N} \to \mathbb{R}_+ \), \hspace{1cm} i = 1, 2, \ldots, n,

(ii) for each \( B \in \mathbb{C}_x^{N,N} \), \( F_i(B) \) depends only on the products

\[
m(B_{k,k})M(B_{k,j}^{-1}B_{j,i}), \hspace{1cm} k, j = 1, 2, \ldots, n, \hspace{1cm} k \neq j,
\]

i.e. if \( C \in \mathbb{C}_x^{N,N} \) with \( m(C_{k,k})M(C_{k,j}^{-1}C_{j,i}) = m(B_{k,k})M(B_{k,j}^{-1}B_{j,i}) \) for all \( k, j = 1, 2, \ldots, n, \hspace{1cm} k \neq j \), then \( F_i(C) = F_i(B), \hspace{1cm} i = 1, 2, \ldots, n \).

We remark that in the case \( N = n \), so that each \( W_i, \hspace{1cm} i = 1, 2, \ldots, n \), is, according to our previous assumption, generated by the single vector \( e_i \), it can be verified that for any \( B = (b_{i,j}) = (B_{i,j}) \in \mathbb{C}_x^{N,N} \),

\[
m(B_{i,j})M([B_{i,j}]^{-1}B_{j,i}) = |b_{i,j}|(\psi_i(e_j)/\psi_j(e_i)), \hspace{1cm} i, j = 1, 2, \ldots, n, \hspace{1cm} i \neq j.
\]

From this, it follows that any \( F \in \mathcal{F}_x \) in this case can be trivially extended to a \( f \in \mathcal{F}_n \).

This brings us to our fundamental definition.

**Definition.** Given \( F = (F_1, F_2, \ldots, F_n) \in \mathcal{F}_n \), then \( F \) is a G-function (relative to \( \pi \)), if, for each \( B = (B_{i,j}) \in \mathbb{C}_x^{N,N} \) for which

\[
m(B_{i,i}) > F_i(B), \hspace{1cm} i = 1, 2, \ldots, n,
\]

(2.1)
$B$ is nonsingular. The collection of all $G$-functions relative to $\pi$ is denoted by $\mathcal{G}_\pi$.

If $F \in \mathcal{G}_\pi$ and if $B = (B_{i,j}) \in C_{n \times n}$, then $\mathcal{M}_\pi^F(B) = (\beta_{i,j}) \in C^{n \times n}$ is defined as the $n \times n$ matrix whose entries are

$$\beta_{i,j} = -m(B_{i,j})M(B_{i,j}^{-1}B_{i,j}) \quad \text{for all} \quad i \neq j, \quad \beta_{i,i} = F_i(B), \quad i, j = 1, 2, \ldots, n.
$$

(2.2)

In analogy with Proposition 1, we now establish the following theorem.

**Theorem 1.** Let $F \in \mathcal{G}_\pi$. Then $F \in \mathcal{G}_\pi$ if and only if $\mathcal{M}_\pi^F(B)$ is a (possibly singular) $M$-matrix for every $B \in C_{n \times n}$.

**Proof.** First, assume that $\mathcal{M}_\pi^F(B)$ is an $M$-matrix for every $B \in C_{n \times n}$.

If $B = (B_{i,j}) \in C_{n \times n}^N$ satisfies (2.1), set $0 < \delta_i = m(B_{i,i}) - F_i(B), \ i = 1, 2, \ldots, n$, so that $m(B_{i,i}) = F_i(B) + \delta_i$. By our definition, $\mathcal{M}_\pi^F(B) + \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ is a nonsingular $M$-matrix, with diagonal entries $m(B_{i,i})$ and off-diagonal entries $-m(B_{i,j})M(B_{i,j}^{-1}B_{i,j})$. But then, $B$ is nonsingular (cf. Fiedler and Pták [3, Theorem 3.3], and Robert [9, Theorem 7]), and we evidently have that $F \in \mathcal{G}_\pi$.

Conversely, assume $F \in \mathcal{G}_\pi$, and consider the matrix $\mathcal{M}_\pi^F(B)$ for any $B \in C_{n \times n}$. From the definition of (2.2), the entries of $\mathcal{M}_\pi^F(B)$ have the proper sign-pattern for $\mathcal{M}_\pi^F(B)$ to be an $M$-matrix. If, on the contrary, $\mathcal{M}_\pi^F(B)$ is not an $M$-matrix, there necessarily exist $\delta_i > 0, \ i = 1, 2, \ldots, n$, such that the matrix $T \equiv \mathcal{M}_\pi^F(B) + \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ is singular, and hence there exists $0 \neq y \in C^n$ for which $Ty = 0$. Equivalently, $Ty = 0$ can be expressed as

$$\begin{align*}
(F_i(B) + \delta_i)y_i - \sum_{j \neq i}^n m(B_{i,j})M(B_{i,j}^{-1}B_{i,j})y_j &= 0, \quad i = 1, 2, \ldots, n. 
\end{align*}
$$

(2.3)

Now, let $\xi_i \in W_i$ be fixed vectors with $\phi_i(\xi_i) = 1, \ i = 1, 2, \ldots, n$, and set $z = (\sum_{i=1}^n \gamma_i \xi_i) \in C^n$. Clearly, $z \neq 0$. We now construct a matrix $C = (C_{i,j}) \in C_{n \times n}$ such that $Cz = 0$, and such that

$$m(C_{i,j})M(C_{i,j}^{-1}C_{i,j}) = m(B_{i,j})M(B_{i,j}^{-1}B_{i,j}), \quad i, j = 1, 2, \ldots, n, \quad i \neq j.
$$

(2.4)

First, if we choose

$$C_{i,i} = (F_i(B) + \delta_i)I_i, \quad i = 1, 2, \ldots, n,
$$

(2.5)
where \( I \) is the identity operator on \( W \), then each \( C_{i,j} \) is evidently nonsingular, so that \( C = (C_{i,j}) \in C_{n,n}^{N,N} \). Moreover, this choice reduces (2.4) to

\[
M(C_{i,j}) = m(B_{i,j})M(B_{i,j}^{-1}B_{i,j}), \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{2.6}
\]

Next, we directly verify from (2.3) and (2.5) that \( Cz = 0 \) if

\[
C_{i,j}x_j = -m(B_{i,j})M(B_{i,j}^{-1}B_{i,j})x_i, \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{2.7}
\]

Our problem then reduces to constructing submatrices \( C_{i,j} : W_j \to W_i \), for all \( i, j = 1, 2, \ldots, n, \quad i \neq j \), which simultaneously satisfy (2.6) and (2.7). Following the construction of Johnston [7], let \( \bar{y}_j \) denote the conjugate norm to \( y_i \) in \( W_i \). As a well-known consequence of the Hahn-Banach Theorem, there exists a vector \( \sigma_j \) in \( W_j \) for which both \( \bar{y}_j(\sigma_j) = 1 \) and \( \sigma_j^*x_i = 1 \). Then, upon defining the submatrices \( C_{i,j} : W_j \to W_i, \quad i \neq j \), by

\[
C_{i,j} = -m(B_{i,j})M(B_{i,j}^{-1}B_{i,j})x_i \sigma_j^*, \quad i, j = 1, 2, \ldots, n, \quad i \neq j,
\]

it is readily seen that both (2.6) and (2.7) are satisfied. In summary, the above construction gives us from (2.7) that \( Cz = 0 \), so that \( C \) is singular. On the other hand, because of (2.4) we see that

\[
F_i(C) = F_i(B), \quad i = 1, 2, \ldots, n,
\]

and from (2.5) that

\[
m(C_{i,i}) = F_i(B) + d_i > F_i(B) = F_i(C), \quad i = 1, 2, \ldots, n.
\]

But, as \( F \in \mathcal{G}_a \) by assumption, the above inequalities give us that \( C \) must be nonsingular, a contradiction. Thus, \( \mathcal{M}_n^F(B) \) is an \( M \)-matrix for each \( B \in C_{n,n}^{N,N} \), and the proof is complete. \( \square \)

3. THE IDENTIFICATION OF \( \mathcal{P}_n \) AND \( \mathcal{P}_x \)

We now show that there is a natural identification of elements of \( \mathcal{P}_n \) and \( \mathcal{P}_x \), given \( n, N(\geq n) \), and \( x \). First, given \( f = (f_1, f_2, \ldots, f_n) \in \mathcal{P}_n \), we define the mapping \( \phi : \mathcal{P}_n \to \mathcal{P}_x \) by \( \phi(f) = (\phi_1(f), \phi_2(f), \ldots, \phi_n(f)) \) where

\[
(\phi_i(f))(B) = f_i(\mathcal{M}_n^F(B)), \quad i = 1, 2, \ldots, n, \quad \text{for all } B \in C_{n,n}^{N,N}, \tag{3.1}
\]

where \( Z \) denotes the zero function in \( \mathcal{P}_x \). Conversely, to define the mapping \( \chi : \mathcal{P}_x \to \mathcal{P}_n \), fix, for each ordered pair \( (i, j) \), \( i, j = 1, 2, \ldots, n, \quad i \neq j \), a submatrix \( E_{i,j} : W_j \to W_i \) for which \( M(E_{i,j}) = 1 \). Then, given \( A = (a_{i,j}) \in C_{n,n} \), define \( B^A = (B_{i,j}^A) \in C_{n,n}^{N,N} \) by
\[ B_{i,j}^4 = |a_{i,j}|E_{i,j}, \quad i \neq j, \quad B_{i,i}^4 = I_i, \quad i, j = 1, 2, \ldots, n. \]

Note that $B^4$ depends only on the moduli of the off-diagonal entries of $A$, and, moreover, that $m(B_{i,j}^4)M[(B_{i,j}^4)^{-1}B_{i,j}^4] = |a_{i,j}|$ for all $i \neq j$. Then, given $F = (F_1, F_2, \ldots, F_n) \in \mathcal{P}_n$, we define the mapping $\chi: \mathcal{P}_n \rightarrow \mathcal{P}_n$ by $\chi(F) = (\chi_1(F), \chi_2(F), \ldots, \chi_n(F))$ where

\[ (\chi_i(F))(A) = F_i(B^4), \quad i = 1, 2, \ldots, n, \quad \text{for all } A \in \mathbb{C}^{n \times n}. \quad (3.2) \]

With these definitions of $\phi$ and $\chi$, we see that the composition $\chi \circ \phi$ maps $\mathcal{P}_n$ into $\mathcal{P}_n$. More precisely, given any $f = (f_1, f_2, \ldots, f_n) \in \mathcal{P}_n$, then $\langle \chi \circ \phi \rangle(f) \equiv (g_1, g_2, \ldots, g_n) \in \mathcal{P}_n$ is given by

\[ g_i(A) = f_i(\mathcal{M}^2(B^4)), \quad i = 1, 2, \ldots, n, \quad \text{for all } A \in \mathbb{C}^{n \times n}, \]

from which it readily follows that

\[ (\chi \circ \phi)(f) = f \quad \text{for all } f \in \mathcal{P}_n. \]

Similarly, $\phi \circ \chi: \mathcal{P}_n \rightarrow \mathcal{P}_n$, and given any $F = (F_1, F_2, \ldots, F_n) \in \mathcal{P}_n$, then $\langle \phi \circ \chi \rangle(F) \equiv (G_1, G_2, \ldots, G_n) \in \mathcal{P}_n$ is given by

\[ G_i(A) = F_i(B_\mathcal{M}^2(A)), \quad i = 1, 2, \ldots, n, \quad \text{for all } A \in \mathbb{C}^{n \times n}, \]

from which analogously it follows that

\[ (\phi \circ \chi)(F) = F \quad \text{for all } F \in \mathcal{P}_n. \]

Thus, using these mappings $\phi$ and $\chi$, we can identify elements of $\mathcal{P}_n$ with elements of $\mathcal{P}_n$, and conversely. Moreover, this identification preserves certain other properties as well in $\mathcal{P}_n$ and $\mathcal{P}_n$, as we show in Theorem 2 below. For added notation, we first define a partial order on $\mathcal{P}_n$. If $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$ are in $\mathcal{P}_n$, we write $f \succeq g$ if

\[ f_i(A) \succeq g_i(A) \quad \text{for all } i = 1, 2, \ldots, n, \quad A \in \mathbb{C}^{n \times n}. \]

The analogous partial order is then used for $\mathcal{P}_n$. Next, we say that $f = (f_1, f_2, \ldots, f_n)$ in $\mathcal{P}_n$ is continuous if, for each $i = 1, 2, \ldots, n$, $f_i$ is continuous on $\mathbb{C}^{n \times n}$. Similarly, $F = (F_1, F_2, \ldots, F_n)$ in $\mathcal{P}_n$ is continuous if, for each $i = 1, 2, \ldots, n, F_i$ is continuous on $\mathbb{C}^{N \times N}$. Because $F_i(B)$, for $B = (B_{i,j}) \in \mathbb{C}^{N \times N}$, by definition depends only on the $n(n - 1)$ products:

\[ a_{i,j} = m(B_{i,j})M[(B_{i,j})^{-1}B_{i,j}], \quad i, j = 1, 2, \ldots, n, \quad i \neq j, \]

then $F \in \mathcal{P}_n$ is continuous if and only if each $F_i$ is continuous with respect to the $n(n - 1)$ quantities $a_{i,j}$. This brings us to Theorem 2.
Theorem 2. Given $n$, $N$, and $x$, the mappings $\phi: \mathcal{P}_n \to \mathcal{P}_n$ and $\chi: \mathcal{P}_n \to \mathcal{P}_n$ are inverses of one another, i.e., given $f \in \mathcal{P}_n$ and $F \in \mathcal{P}_n$, then $\phi(f) = F$ if and only if $\chi(F) = f$. These mappings preserve:

(i) nonnegative linear combinations, i.e., for arbitrary scalars $r \geq 0$, $s \geq 0$,

$$\phi(rf + sg) = r\phi(f) + s\phi(g), \quad \text{for all } f, g \in \mathcal{P}_n,$$

$$\chi(rF + sG) = r\chi(F) + s\chi(G), \quad \text{for all } F, G \in \mathcal{P}_n;$$

(ii) partial order, i.e., for $f = \chi(F)$, $g = \chi(G)$ in $\mathcal{P}_n$, $F = \phi(f)$, $G = \phi(g)$ in $\mathcal{P}_n$, then $f \leq g$ if and only if $F \leq G$;

(iii) continuity, i.e., $f = \chi(F) \in \mathcal{P}_n$ is continuous if and only if $F = \phi(f) \in \mathcal{P}_n$ is continuous;

(iv) $G$-functions, i.e., $f = \chi(F) \in \mathcal{P}_n$ is a $G$-function if and only if $F = \phi(f) \in \mathcal{P}_n$ is a $G$-function (relative to $x$).

Proof. The proofs of (i)–(iii) are readily verified, and are omitted. To prove (iv), we make use of the identity,

$$\mathcal{M}_{\chi}(B) = \mathcal{M}_{\chi}(\mathcal{M}_{\phi}(B)) \quad \text{for all } B \in \mathbb{C}_{n}^{N,N}, \quad \text{all } f \in \mathcal{P}_n, \quad (3.3)$$

which follows directly from (3.1). If $f \in \mathcal{G}_n$, then from Proposition 1 and the above identity, $\mathcal{M}_{\chi}(B)$ is evidently an $M$-matrix for every $B \in \mathbb{C}_{n}^{N,N}$.

Thus, from Theorem 1, $\phi(f) \in \mathcal{G}_n$.

Conversely, making use of the analogous identity (cf. (3.2)),

$$\mathcal{M}_{\phi}(F)(A) = \mathcal{M}_{\phi}(FA) \quad \text{for all } A \in \mathbb{C}^{n,n}, \quad \text{all } F \in \mathcal{P}_n, \quad (3.4)$$

assume that $F \in \mathcal{G}_n$. Thus, from Theorem 1 again, $\mathcal{M}_{\phi}(F)(A)$ is an $M$-matrix for every $A \in \mathbb{C}^{n,n}$, so that from Proposition 1, $\chi(F) \in \mathcal{G}_n$. This completes the proof.

We now say that $B \in \mathbb{C}_{x}^{N,N}$ is irreducible, if the associated matrix $\mathcal{M}_{\chi}(B) \in \mathbb{C}^{n,n}$ is irreducible in the usual graph-theoretic sense (cf. Varga [10, p. 19]). Note that for $B \in \mathbb{C}_{x}^{N,N}$, irreducibility is equivalent to the notion of block-irreducibility of Feingold and Varga [2].

Using Theorem 2 and the above notion of irreducibility, many of the results of Carlson and Varga [1] directly carry over the partitioned matrix case. We give several such results now as corollaries of Theorem 2.

Corollary 1. If $F \in \mathcal{G}_n$ and if $B \in \mathbb{C}_{x}^{N,N}$ is irreducible, then there is an $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ with $x > 0$ (depending on $B$) such that
\[ F_i(B) \geq R^*(B) = \frac{1}{x_i} \sum_{j \neq i}^n m(B_{i,j})M(B_{i,j}^{-1}B_{j,i})x_j, \quad i = 1, 2, \ldots, n. \quad (3.5) \]

Moreover, if \( \mathcal{M}_n^F(B) \) is a singular \( M \)-matrix, equality holds throughout above, i.e., \( F_i(B) = R^*(B), \quad i = 1, 2, \ldots, n. \)

**Proof.** We include the proof of this corollary to show how our previous constructions and the results of Theorem 2 couple with the results of [1]. If \( B \in \mathbb{C}_x^{N,N} \) is irreducible, the \( n \times n \) matrix \( A = (a_{i,j}) = \mathcal{M}_n^F(B) \) is, by definition, irreducible. Assuming \( F \in \mathcal{G}_n \), then \( f = \chi(F) \) is a \( G \)-function from (iv) of Theorem 2. Thus, from Proposition 1 of [1], there is an \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \) with \( x > 0 \) (depending on \( A \), and hence depending on \( B \)) such that

\[ f_i(A) \geq r^*(A) = \frac{1}{x_i} \sum_{j \neq i}^n |a_{i,j}|x_j, \quad i = 1, 2, \ldots, n. \]

But, as \( \phi(f) = F \), it follows from (3.1) and the definition of the entries of the matrix \( A \) that the above inequalities turn out to be nothing more than

\[ F_i(B) \geq \frac{1}{x_i} \sum_{j \neq i}^n m(B_{i,j})M(B_{i,j}^{-1}B_{j,i})x_j, \quad i = 1, 2, \ldots, n, \]

the desired result of (3.5). The case of equality in (3.5) similarly follows, which completes the proof. \( \blacksquare \)

In analogy with [1], we denote by \( \mathcal{G}_n^e \) and \( \mathcal{G}_n^{e*} \), respectively, the sets of continuous functions in \( \mathcal{G}_n \) and \( \mathcal{G}_n^* \). The next corollary follows directly from Hoffman's result [4] (cf. [1, Theorem 1]) and Theorem 2.

**Corollary 2.** If \( F = (F_1, F_2, \ldots, F_n) \) and \( G = (G_1, G_2, \ldots, G_n) \) are in \( \mathcal{G}_n \) and \( 0 < \alpha < 1 \), then \( H = (H_1, H_2, \ldots, H_n) \) and \( K = (K_1, K_2, \ldots, K_n) \), defined by

\[ H_i(B) = F_i^*(B)G_i^{1-\alpha}(B), \quad i = 1, 2, \ldots, n, \quad \text{for all } B \in \mathbb{C}_x^{N,N}, \quad (3.6) \]

\[ K_i(B) = \alpha F_i(B) + (1 - \alpha)G_i(B), \quad i = 1, 2, \ldots, n, \quad \text{for all } B \in \mathbb{C}_x^{N,N}, \quad (3.7) \]
are also in $\mathcal{G}_n$. If $F$ and $G$ are in $\mathcal{G}_n^\varepsilon$, so are $H$ and $K$. Thus, $\mathcal{G}_n$ and $\mathcal{G}_n^\varepsilon$ are convex sets.

Because of the partial order and convexity that exist in $\mathcal{G}_n$, we say, in analogy with [1], that $F \in \mathcal{G}_n(\mathcal{G}_n^\varepsilon)$ is minimal in $\mathcal{G}_n(\mathcal{G}_n^\varepsilon)$ if, for every $G \in \mathcal{G}_n(\mathcal{G}_n^\varepsilon)$ for which $G \leq F$, we have $G = F$. Similarly, we say that $F \in \mathcal{G}_n(\mathcal{G}_n^\varepsilon)$ is an extreme point of the convex set $\mathcal{G}_n(\mathcal{G}_n^\varepsilon)$ if $F = \alpha G + (1 - \alpha)H$, where $0 < \alpha < 1$ and where $G$ and $H$ are in $\mathcal{G}_n(\mathcal{G}_n^\varepsilon)$, implies that $F = G = H$. The proof given in [1] then shows that the minimal elements of $\mathcal{G}_n(\mathcal{G}_n^\varepsilon)$ are precisely the extreme points of $\mathcal{G}_n(\mathcal{G}_n^\varepsilon)$. Other characterizations of minimal elements in $\mathcal{G}_n^\varepsilon$ follow directly from Corollary 1 and from the analogous results of [1, Theorem 2], which we state as Theorem 3 below.

**Theorem 3.** Let $F \in \mathcal{G}_n^\varepsilon$. Then the following are equivalent:

(i) $F$ is minimal in $\mathcal{G}_n^\varepsilon$;
(ii) $F$ is an extreme point in $\mathcal{G}_n^\varepsilon$;
(iii) for every $B \in \mathcal{C}_n^{N,N}$, the matrix $M_F(B)$ is singular;
(iv) for every irreducible $B \in \mathcal{C}_n^{N,N}$, there exists an $x \in \mathbb{C}^n$ with $x > 0$ (depending on $B$) for which (cf. (3.5))

$$F_i(B) = R_i^x(B), \quad i = 1, 2, \ldots, n.$$ 

In a similar way, it is easily verified from the proof of [1, Theorem 4] that the following result is also valid.

**Theorem 4.** For $n > 2$, let $F$ and $G$ be two distinct minimal elements in $\mathcal{G}_n^\varepsilon$, and for $0 < \alpha < 1$, let $H \in \mathcal{G}_n^\varepsilon$ be defined by (3.6) of Corollary 2. Then, $H$ is not minimal in $\mathcal{G}_n^\varepsilon$.

The reader will readily see that other results from [1], such as $(\alpha, \beta)$-convolutions of elements in $\mathcal{G}_n$, the characterization [1, Theorem 6] of the minimal elements in $\mathcal{G}_n$, which allows both for discontinuous elements in $\mathcal{G}_n$ and reducible matrices in $\mathbb{C}^{n,n}$, and results on domains of dependence (cf. [5]), carry over directly to the partitioned case. For brevity, we have omitted these extensions.

As a final note, we remark that the above analysis could just as well have been carried out by using $M(B_{i,j})$ throughout in place of $m(B_{i,j}) \cdot M(B_{i,j}^{-1}B_{i,j}), i = 1, 2, \ldots, n, i \neq j$, and by defining the elements of $\mathcal{P}_n$ as functions from $\mathbb{C}^{N,N}$ (instead of $\mathbb{C}_n^{N,N}$) to $\mathbb{R}^{n^2}$. While this would have
given a more direct "off-diagonal" analogue of the results of [1], it is however known from the results of Fiedler and Pták [3] and Robert [9] that, for fixed vector norms \( \psi_k \) on \( W_i \), \( i = 1, 2, \ldots, n \), and for an arbitrary \( B_i, j: W_j \to W_i \), and for an arbitrary nonsingular \( B_i, i: W_i \to W_i \),

\[
m(B_i, i) M(B^{-1}_i, i) \leq M(B_i, i), \quad i \neq j.
\]

This means that the Gerschgorin sets defined by

\[
\{ \varepsilon \in C: m(B_{i, i} - \varepsilon I) \leq F_i(B) \}, \quad i = 1, 2, \ldots, n, \tag{3.8}
\]

where \( F = (F_1, F_2, \ldots, F_n) \in \mathcal{A}_n \) depends on the products \( m(B_{i, i}) \cdot M(B^{-1}_i, i) \), \( i \neq j \), will be smaller than the analogous sets with \( F_i(B) \) depending on the numbers \( M(B_{i, i}) \), \( i \neq j \), if each \( F_i \) is a monotone non-decreasing function of its argument. (Note that the row sums \( R^\pi \) of (3.5) have this property.) Consequently, the union of the sets of (3.5) will determine a smaller region in the complex plane which contains all the eigenvalues of \( B \) which are not also eigenvalues of some \( B_i, i \) (cf. [6]), than that produced analogously by the \( F_i(B) \) depending on the numbers \( M(B_{i, i}) \).

REFERENCES


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