90

On Recurring Theorems on Diagonal Dominance

Dedicated to Olga Taussky Todd

Richard S. Varga†
Department of Mathematics
Kent State University
Kent, Ohio 44242

Submitted by David Carlson

ABSTRACT

The elementary, but very useful, concept of a strictly diagonally dominant $n \times n$ complex matrix has seen various generalizations over the course of years. The primary purpose of this note is to give a number of equivalent conditions which reproduce, and in some cases strengthen, many consequences of recent generalizations of the property of diagonal dominance.

1. INTRODUCTION

In this day and age, students learn early in their training that a strictly diagonally dominant $n \times n$ complex matrix is necessarily nonsingular. Possibly because this concept is so easily grasped, and also possibly because this concept is so useful in many applications, many generalizations of strict diagonal dominance have appeared, and many continue to appear. It seems, however, less well known that most of these generalizations can in fact be based on the theory of M-matrices, introduced by Ostrowski [16]. In view of this, the object of this note, like that of the famous paper by Olga Taussky [19] on "A recurring theorem on determinants", is to establish equivalences between various related recent results in this area. In so doing, some known results are reproduced here, and some possibly less well known connections are also strengthened here.

2. NOTATION

Let $\mathbf{C}^{n,n}$ denote the collection of all $n \times n$ complex matrices $A = [a_{i,j}]$, and let $\mathbf{C}^{n,n}_{\pi}$ denote the subset of all $n \times n$ complex matrices with all diagonal

†Research supported in part by the Energy Research and Development Administration (ERDA) under Grant ET(11-1)-2075, and by the U. S. Air Force under Grant AFOSR 74-2729.

entries nonzero. Similarly, $\mathbf{C}^n$ denotes the complex n-dimensional vector space of all column vectors $\mathrm{v}=[v_1,v_2,\ldots,v_n]^T$ where each component v_i is a complex number, i.e., $v_i\in\mathbf{C}$. The restriction to real entries or components similarly defines $\mathbf{R}^{n,n}$, $\mathbf{R}^{n,n}_{\pi}$, and $\mathbf{R}^n$. Next, if $A\in\mathbf{C}^{n,n}$, its *spectral radius*, $\rho(A)$, is defined as usual by $\rho(A)=\max\{|\lambda|:\det(\lambda I-A)=0\}$. Also, it is convenient to set $N\equiv\{1,2,\ldots,n\}$, and, for $\mathrm{v}\in\mathbf{R}^n$, we write $\mathrm{v}>\mathbf{0}$ or $\mathrm{v}\geqslant\mathbf{0}$ if $v_i>0$ or $v_i\geqslant 0$ for all $i\in N$. Similarly, for $A=[a_{i,j}]\in\mathbf{R}^{n,n}$, we write $A>\mathcal{O}$ or $A\geqslant\mathcal{O}$ if $a_{i,j}>0$ or $a_{i,j}\geqslant 0$ for all $i,j\in N$. For $A,B\in\mathbf{R}^{n,n}$, we also write $A\geqslant B$ if $(A-B)\geqslant\mathcal{O}$. Next, for $A=[a_{i,j}]\in\mathbf{C}^{n,n}$, we define $|A|\equiv[|a_{i,j}|]\in\mathbf{R}^{n,n}$, and

$$\|A\|_\infty\equiv\max_{i\in N}\left\{\sum_{j\in N}|a_{i,j}|\right\}. \tag{2.1}$$

A matrix $A=[a_{i,j}]\in\mathbf{C}^{n,n}$ is said to be *reducible* if there exists a nonempty subset $S\subset N$ with $S\neq N$, such that $a_{i,j}=0$ for all $i\in S$ and all $j\in N\backslash S$. A matrix is said to be *irreducible* if it is not reducible. As is well known (cf. [20, p. 20]), a matrix is irreducible if and only if its directed graph is strongly connected.

Given any $A=[a_{i,j}]\in\mathbf{C}^{n,n}$, we define its *comparison matrix* $\mathfrak{M}(A)=[\alpha_{i,j}]\in\mathbf{R}^{n,n}$ by

$$\alpha_{i,i}=|a_{i,i}|;\qquad \alpha_{i,j}=-|a_{i,j}|,\qquad i\neq j;\quad i,j\in N, \tag{2.2}$$

and we define

$$\Omega(A)\equiv\{B=[b_{i,j}]\in\mathbf{C}^{n,n}:|b_{i,j}|=|a_{i,j}|,\ i,j\in N\}, \tag{2.3}$$

the so-called *equimodular set of matrices*, associated with A. Note that both A and $\mathfrak{M}(A)$ are elements of $\Omega(A)$.

Next, given any $A=[a_{i,j}]\in\mathbf{C}^{n,n}_{\pi}$, we can decompose each $B=[b_{i,j}]\in\Omega(A)$ into the sum

$$B=D(B)-L(B)-U(B), \tag{2.4}$$

where $D(B)\equiv\mathrm{diag}[b_{1,1},b_{2,2},\ldots,b_{n,n}]$, and where $L(B)$ and $U(B)$ are respectively strictly lower and strictly upper triangular matrices. Then, define, for $\omega>0$,

$$J_\omega(B)\equiv\omega[D(B)]^{-1}[L(B)+U(B)]+(1-\omega)I=\omega J_1(B)+(1-\omega)I, \tag{2.5}$$

and

$$\mathfrak{L}_\omega(B)\equiv[D(B)-\omega L(B)]^{-1}[(1-\omega)D(B)+\omega U(B)], \tag{2.6}$$

to be respectively the (point) *Jacobi overrelaxation iteration matrix*, and the (point) *successive overrelaxation iteration matrix* associated with B. For

$\omega = 1$, note that (2.5) and (2.6) reduce respectively to the familiar (point) Jacobi and the (point) Gauss-Seidel iterative methods.

As is well known, $A = [a_{i,j}] \in \mathbf{C}_{\pi}^{n,n}$ is said to be *strictly diagonally dominant* if

$$|a_{i,i}| > \sum_{\substack{j \in N \\ j \neq i}} |a_{i,j}|, \qquad i \in N, \tag{2.7}$$

and *irreducibly diagonally dominant* if A is irreducible and if

$$|a_{i,i}| \geqslant \sum_{\substack{j \in N \\ j \neq i}} |a_{i,j}|, \qquad i \in N, \tag{2.8}$$

with strict inequality holding in (2.8) for at least one $i \in N$. The above matrix concepts can be simultaneously generalized as follows. Consider any $B = [b_{i,j}] \in \mathbf{R}^{n,n}$ for which $b_{i,j} \leqslant 0$ for all $i \neq j$; $i,j \in N$. Extending slightly an earlier definition by Bramble and Hubbard [4], we say that B is of *generalized positive type* if there exists a $\mathbf{u} \in \mathbf{R}^n$ such that

$$\left.\begin{array}{l} \text{(i) } \mathbf{u} > \mathbf{0},\ B\mathbf{u} \geqslant \mathbf{0}, \text{ and } \{i \in N : (B\mathbf{u})_i > 0\} \text{ is nonempty;} \\ \text{(ii) for each } i_0 \in N \text{ with } (B\mathbf{u})_{i_0} = 0, \text{ there exist indices} \\ \quad i_1, i_2, \ldots, i_r \text{ in } N \text{ with } b_{i_k, i_{k+1}} \neq 0,\ 0 \leqslant k \leqslant r-1, \\ \quad \text{such that } (B\mathbf{u})_{i_r} > 0. \end{array}\right\} \tag{2.9}$$

With the notation of (2.2), note that if $A = [a_{i,j}] \in \mathbf{C}^{n,n}$ is either strictly or irreducibly diagonally dominant, then $\mathfrak{M}(A)$ is of generalized positive type, since (2.9) is satisfied by $B = \mathfrak{M}(A)$ and $\mathbf{u} = \boldsymbol{\zeta} \equiv [1, 1, \ldots, 1]^T$. We remark that the original definition of Bramble and Hubbard [4] was made with respect to the fixed vector $\boldsymbol{\zeta}$.

Finally, any matrix $B = [b_{i,j}] \in \mathbf{R}^{n,n}$ with $b_{i,j} \leqslant 0$ for all $i \neq j$; $i,j \in N$, can be written as

$$B = \tau I - C \tag{2.10}$$

where $\tau \equiv \max_{i \in N} b_{i,i}$, and where $C = [c_{i,j}] \in \mathbf{R}^{n,n}$, satisfying $C \geqslant \mathbb{O}$, has its entries defined by

$$c_{i,i} = \tau - b_{i,i} \geqslant 0, \qquad c_{i,j} = -b_{i,j} \geqslant 0, \qquad i \neq j; \quad i,j \in N. \tag{2.11}$$

Following Ostrowski [16], such a matrix B is said to be a *nonsingular M-matrix* if $\tau > \rho(C)$.

With these notations, we now give our equivalence theorem.

3. EQUIVALENCE THEOREM

THEOREM 1. *For any* $A=[a_{i,j}]\in \mathbf{C}_{\pi}^{n,n}$, *the following are equivalent:*

(i) $\mathfrak{M}(A)$, *defined in* (2.2), *is a nonsingular M-matrix;*
(ii) *the real part of each eigenvalue of* $\mathfrak{M}(A)$ *is positive;*
(iii) *there exists a* $\mathbf{u}\in\mathbf{R}^n$ *with* $\mathbf{u}>\mathbf{0}$ *such that* $\mathfrak{M}(A)\cdot\mathbf{u}>\mathbf{0}$;
(iv) $\mathfrak{M}(A)$ *is of generalized positive type* [*cf.* (2.9)];
(v) *there exists a* $\mathbf{u}\in\mathbf{R}^n$ *with* $\mathbf{u}>\mathbf{0}$ *such that* $\mathcal{M}(A)\cdot\mathbf{u}\geq\mathbf{0}$, *and such that* $\sum_{j\leq i}\alpha_{i,j}u_j>0$ *for all* $i\in N$, *where* $\mathcal{M}(A)\equiv[\alpha_{i,j}]$;
(vi) *there exist upper and lower triangular nonsingular M-matrices L and U such that* $\mathcal{M}(A)=L\cdot U$;
(vii) *for any* $B\in\Omega(A)$, $\rho(J_1(B))\leq\rho(|J_1(B)|)=\rho(J_1(\mathfrak{M}(A)))<1$;
(viii) *for any* $B\in\Omega(A)$ *and for any* $0<\omega<2/[1+\rho(J_1(B))]$,

$$\rho(J_\omega(B))\leq\omega\rho(J_1(B))+|1-\omega|<1; \tag{3.1}$$

(ix) *for any* $B\in\Omega(A)$ *and for any* $0<\omega<2/[1+\rho(|J_1(B)|)]$,

$$\rho(\mathcal{L}_\omega(B))\leq\omega\rho(|J_1(B)|)+|1-\omega|<1. \tag{3.2}$$

Finally, for any $A\in\mathbf{C}^{n,n}$, *the following are equivalent:*
(x) *each* $B\in\Omega(A)$ *is nonsingular;*
(xi) *there exists a (unique) permutation matrix* $P\in\mathbf{R}^{n,n}$ *such that* $\mathfrak{M}(PA)$ *is a nonsingular M-matrix.*

Proof. That (i), (ii), and (iii) are equivalent is well known; see, for example, Fan [9]. See also Fiedler and Pták [10], where numerous additional equivalences are given. We remark that, in the notation of Ostrowski [16], any matrix satisfying (i) of Theorem 1 is by definition an *H-matrix*.

Continuing, that (iii) implies (iv) is trivial. Conversely, assuming (iv), i.e., that $\mathcal{M}(A)$ is of generalized positive type, the original proof of Bramble and Hubbard [4] shows that $\mathcal{M}(A)$ is a nonsingular M-matrix, whence (iv) implies (i). See also Bohl [3, Satz 3.2]. Thus, the first four properties of Theorem 1 are equivalent. Next, because of the signs of the entries of $\mathcal{M}(A)$, (iii) directly implies (v). Conversely, (v) implies (iv), i.e., that $\mathcal{M}(A)$ is of generalized positive type. To see this, assume (v) and note that the choice $i=n$ in (v) gives that $(\mathcal{M}(A)\cdot\mathbf{u})_n>0$, whence (2.9i) holds for $\mathcal{M}(A)$. If $\mathcal{M}(A)$ is irreducible (the reducible case being similar), then (2.9ii) holds, so that $\mathcal{M}(A)$ is of the generalized positive type. Thus, the first five properties of Theorem 1 are equivalent. That (vi) is equivalent to, say, (iii) is due to Fiedler and Pták [10, Thm. 4.3; No. 10]. (We remark here that

Fiedler and Pták give (vi) in a slightly weaker form involving an unnecessary permutation transformation, but their technique of proof applied directly to (vi).)

For any $B=[b_{i,j}]\in\Omega(A)$, the entries of $J_1(B)=[k_{i,j}]$ satisfy, from (2.3) and (2.5),

$$k_{i,i}=0, \qquad |k_{i,j}|=\frac{|a_{i,j}|}{|a_{i,i}|}, \qquad i\neq j; \quad i,j\in N, \tag{3.3}$$

from which we deduce [cf. (2.2)] that

$$|J_1(B)|=J_1(\mathfrak{M}(A)), \qquad B\in\Omega(A). \tag{3.4}$$

From the Perron-Frobenius theory of nonnegative matrices (cf. [20, p. 47]), the above equality yields

$$\rho(J_1(B))\leqslant\rho(|J_1(B)|)=\rho(J_1(\mathfrak{M}(A))), \qquad B\in\Omega(A). \tag{3.5}$$

Then, from [20, Theorems 3.8 and 3.10], it is well known that $\mathfrak{M}(A)$ is a nonsingular M-matrix if and only if $\rho(J_1(\mathfrak{M}(A)))<1$, and thus, with (3.5), (i) and (vii) are equivalent. Hence, the first seven properties of Theorem 1 are equivalent.

Next, from (2.5), we see that, for $\omega>0$,

$$\rho(J_\omega(B))\leqslant\omega\rho(J_1(B))+|1-\omega|, \qquad B\in\Omega(A). \tag{3.6}$$

Assuming (vii), i.e., $\rho(J_1(B))<1$, the bound on the right of (3.6) is seen to be less than unity for any ω with $0<\omega<2/[1+\rho(J_1(B))]$, whence (vii) implies (viii). Again, assuming (vii), i.e., $\rho(|J_1(B)|)<1$, we have from Kulisch [13, *Satz* 3, p. 449] that

$$\rho(\mathcal{L}_\omega(B))\leqslant\omega\rho(|J_1(B)|)+|1-\omega|<1 \tag{3.7}$$

for any $0<\omega<2/[1+\rho(|J_1(B)|)]$, whence (vii) implies (ix).

To complete the equivalence of (i)–(ix), assume (viii) and choose $\mathfrak{M}(A)\in\Omega(A)$, for which we have $\rho(J_1(\mathfrak{M}(A)))=\rho(|J_1(\mathfrak{M}(A))|)$. Next, choose any positive ω less than unity such that $0<\omega<2/[1+\rho(J_1(\mathfrak{M}(A)))]$. Then, from (3.1), $\omega\rho(J_1(\mathfrak{M}(A)))+|1-\omega|<1$ implies that $\omega[\rho(J_1(\mathfrak{M}(A)))-1]<0$. Since $\omega>0$, it follows that $\rho(J_1(\mathfrak{M}(A)))<1$. Hence, with (3.5), (viii) implies (vii). The same proof also shows that (ix) implies (vii), and hence, the first nine properties of Theorem 1 are equivalent.

From the original paper of Ostrowski [16], it is known that each $B\in\Omega(A)$ is nonsingular if $\mathfrak{M}(A)$ is a nonsingular M-matrix. The converse form of this, giving the equivalence of (x) and (xi) in Theorem 1, is due to Camion and Hoffman [5].

We remark that of the equivalences established in Theorem 1, only (v),(viii) and (ix) can be regarded as strengthening known results in the literature.

We now comment on the relation between the equivalences in Theorem 1 and many known recent results in the literature. To begin, James and Riha [12] have recently defined $A=[a_{i,j}]\in\mathbf{C}_{\pi}^{n,n}$ to have *generalized column diagonal dominance* if there exists a $\mathbf{u}=[u_1,u_2,\ldots,u_n]^T\in\mathbf{R}^n$ with $\mathbf{u}>\mathbf{0}$ such that

$$|a_{i,i}|u_i>\sum_{\substack{j\in N\\ j\neq i}}|a_{i,j}|u_j \qquad \text{for all } i\in N. \tag{3.8}$$

This definition, however, is precisely equivalent to (iii) of Theorem 1, i.e., that there exists a $\mathbf{u}\in\mathbf{R}^n$ with $\mathbf{u}>\mathbf{0}$ such that $\mathfrak{M}(A)\cdot\mathbf{u}>\mathbf{0}$. These authors in essence show [12, Theorem 4], under the added (unnecessary) assumption that A is irreducible, that (iii) of Theorem 1 is equivalent to the combined hypotheses [cf. (vii) and (ix) of Theorem 1] that $\rho(J_1(\mathfrak{M}(A)))<1$ and $\rho(\mathcal{L}_\omega(\mathfrak{M}(A)))<1$ for $0<\omega\leqslant 1$. It is also shown in Riha and James [12, Theorem 6] that if $\mathfrak{M}(A)$ is irreducible and symmetric, then $\mathfrak{M}(A)$ is positive definite if and only if A possesses generalized column diagonal dominance [cf. (3.8)]. This follows directly (without the assumption of irreducibility) from the equivalence of (ii) and (iii) in Theorem 1. Similarly James [11, Theorem 1] shows that strict or irreducible diagonal dominance for A, a stronger assumption than (iv) of Theorem 1, implies that $\rho(\mathcal{L}_1(A))<1$, which is weaker than (ix) of Theorem 1. Although the remainder of this paper [11] is concerned with ω's which vary with $i\in N$, it is shown in [11, Corollary to Theorem 2] that if A is strictly diagonally dominant, then $\rho(\mathcal{L}_\omega(A))<1$ if $0<\omega<2/[1+\|J_1(A)\|_\infty]$. Note that since it is well known that $\rho(|J_1(A)|)\leqslant\|J_1(A)\|_\infty$, it follows that $0<\omega<2/[1+\|J_1(A)\|_\infty]$ implies that $0<\omega<2/[1+\rho(|J_1(A)|)]$. Hence the above result follows more generally from the equivalence of (iii) and (ix) of Theorem 1.

Next, given $A=[a_{i,j}]\in\mathbf{C}^{n,n}$, suppose that A is diagonally dominant [cf. (2.9)], i.e., $\mathfrak{M}(A)\cdot\boldsymbol{\zeta}\geqslant\mathbf{0}$. If A is singular, then from (iv) and (x) of Theorem 1, $\mathfrak{M}(A)$ cannot be of generalized positive type with respect to the vector $\boldsymbol{\zeta}$. Without going into detail, we simply remark that negating the property of generalized positive type (with respect to $\boldsymbol{\zeta}$) in (2.9) duplicates the main result of Erdelsky [8, Theorem 1].

In defining his *Zeilensummenbedingung*, Bohl [2,3] weakens the assumptions for generalized positive type matrices in (2.9), by allowing $\mathbf{u}\in\mathbf{R}^n$ in (2.9i) to satisfy $\mathbf{u}\geqslant\mathbf{0}$, but then immediately shows [3, *Satz* 2.1] that if $\mathbf{u}\in\mathbf{R}^n$

with $\mathbf{u} \geqslant \mathbf{0}$ satisfies the remaining conditions of (2.9), then in fact $\mathbf{u} > \mathbf{0}$ and $b_{i,i} > 0$ for all $i \in N$. Consequently, Bohl's *Zeilensummenbedingung* and the hypothesis of generalized positive type are equivalent. For $A \in \mathbf{R}^{n,n}$ with $A \geqslant \mathcal{O}$, Bohl [3, *Satz* 2.2] shows that (i), (iii), and (iv) of Theorem 1 are equivalent when applied to $I - A$.

In Shivakumar and Chew [26], one finds as the main result that the special case (iii) of Theorem 1, with $\mathbf{u} = \zeta$, implies that A is nonsingular, which is weaker than the equivalence of (iii) and (x).

Schäfke [18, *Satz* 1], improving on a paper by Walter [21], gives six equivalent conditions for an $A = [a_{i,j}] \in \mathbf{R}^{n,n}$ satisfying $A \geqslant \mathcal{O}$ and $\|A\|_\infty \leqslant 1$, to have $\rho(A) < 1$. This can be viewed as finding equivalent conditions on A for $I - A$ to be a nonsingular M-matrix. Condition 4 of [18, *Satz* 1], in fact, reduces to (iii) of Theorem 1 in this case.

Next, Kulisch [13, Theorem 1], as a special case, establishes that $\rho(\mathfrak{L}_\omega(B)) < 1$ for any $0 < \omega < 2/[1 + \rho(|J_1(B)|)]$ and for any B with $\rho(|J_1(B)|) < 1$, and deduces [13, Corollary 1.3] that B, being either strictly or irreducibly diagonally dominant, is sufficient for $\rho(|J_1(B)|) < 1$. This last deduction of course follows more generally from the equivalence of (iv) and (vii) of Theorem 1. See also Apostolatos and Kulisch [1].

Continuing, Elsner [7] gives the definition of a "*verallgemeinerte Zeilensummenkriterium*", which turns out to be precisely condition (iii) of Theorem 1, applied to the matrix $I - A$. As consequences of his definition, Elsner in essence shows that (iii) implies (iv) of Theorem 1, and that (iii) implies the convergence of the Gauss-Seidel iterative method, a special case $\omega = 1$ of (ix) of Theorem 1.

Next, Müller [15], in studying the iterative solution of nonlinear systems of equations, formulates the concept [15, Definition 5] of a *chained weakly contractive* system, which, when applied to the linear matrix equation $(I - A)\mathbf{x} = \mathbf{b}$, reduces precisely to condition (iv) of Theorem 1, i.e., $\mathfrak{M}(I - A)$ is of generalized positive type. In the spirit of Theorem 1 and the work of Schäfke [18], Müller [15, *Sätze* 4, 5, 5a] develops ten consequences and equivalences (when $A \geqslant \mathcal{O}$) of his chained weakly contractive system in the linear case—such as that (iv) of Theorem 1 is equivalent to (iii) of Theorem 1, and that (iv) of Theorem 1 is equivalent to $\rho(J_1(\mathfrak{M}(A))) < 1$, as in (vii) of Theorem 1.

In his important study of iterative methods for solving systems of nonlinear equations, Rheinboldt [17, Theorem 4.4] deduces, as a special case, the equivalence of (i), (iii), and (iv) of Theorem 1, and notes that the particular implication—namely, that (iv) for the case $\mathbf{u} = \zeta$ implies (i) in Theorem 1—can be traced back to Duffin [6].

In a similar study of iterative methods for systems of nonlinear equations, Moré [14] gives the definition in the linear case of Ω-*diagonally dominant* matrices, which turns out to be equivalent [cf. (iv) of Theorem 1] to $\mathfrak{M}(A)$ being of generalized positive type with respect to the vector $\mathbf{u}=\zeta=[1,1,\ldots,1]^T$ [cf. (2.9)]. Moré then shows [14, Theorem 4.7] the equivalence of (iv) and (i) of Theorem 1.

To round out our discussion of recent related published results concerning diagonal dominance, in Young's recent book [22], one can deduce [22, p. 43] the equivalence of (i) and (vii) of Theorem 1, and it is shown [22, p. 107] that if A is irreducibly diagonally dominant [a stronger hypothesis than (iv) of Theorem 1], then $\rho(J_\omega(A))<1$ and $\rho(\mathcal{L}_\omega(A))<1$ for any $0<\omega\leqslant 1$ [which are weaker than (viii) and (ix) of Theorem 1]. It is also shown [22, p. 126] that (i) of Theorem 1 implies $\rho(\mathcal{L}_\omega(\mathfrak{M}(A)))<1$ for any $0<\omega<2/[1+\rho(J_1(\mathfrak{M}(A)))]$, which is a special case of (ix).

To conclude, it is interesting to note that Beauwens [23] introduces (v), calling this property (when $\mathbf{u}=\zeta$) *lower semi-strictly diagonally dominance*, and shows in [23] that this property, coupled with irreducibility, is equivalent to irreducible diagonal dominance. Finally, the main result of Jacobsen [24] and Meijerink and Van der Vorst [25] is that a nonsingular symmetric M-matrix (which is necessarily positive definite) can be factored as $G\cdot G^T$ where G is a nonsingular triangular M-matrix, which in essence is weaker than the equivalence of (i) and (vi) of Theorem 1.

The author wishes to express his thanks to Professors Robert Beauwens and David Carlson for helpful additions and comments.

REFERENCES

1 N. Apostolatos and U. Kulisch, Über die Konvergenz des Relaxationsverfahrens bei nicht-negativen und diagonaldominanten Matrizen, *Computing* **2** (1967), 17–24.

2 Erich Bohl, *Monotonie: Lösbarkeit und Numerik bei Operatorgleichungen*, Springer Tracts in Natural Philosophy, **25** (1974).

3 Erich Bohl, Über eine Zeilensummenbedingung bei L-Matrizen, Lecture Notes in Mathematics 395 (Sammelband der Tagung über Numerische Lösung nichtlinearer partieller Differential- und Integrodifferentialgleichungen, in Oberwolfach), Springer, 1974, 247–262.

4 J. H. Bramble and B. E. Hubbard, On a finite difference analogue of an elliptic boundary value problem which is neither diagonally dominant nor of non-negative type, *J. Math. and Phys.* **43** (1964), 117–132.

5 Paul Camion and A. J. Hoffman, On the nonsingularity of complex matrices, *Pac. J. Math.* **17** (1966), 211–214.

6 R. J. Duffin, Nonlinear networks. IIb, *Bull. Am. Math. Soc.* **54** (1948), 119–127.

7 L. Elsner, Bemerkungen zum Zeilensummenkriterium, *Zeit. Angew. Math. Mech.* **49** (1969), 97.

8 P. J. Erdelsky, A general theorem on dominant-diagonal matrices, *Linear Algebra Appl.* **1** (1968), 203–209.

9 K. Fan, Topological proof for certain theorems on matrices with non-negative elements, *Monatsh. Math.* **62** (1958), 219–237.

10 Miroslav Fiedler and Vlastimil Pták, On matrices with non-positive off-diagonal elements and positive principal minors, *Czech. Math. J.* **12** (87) (1962), 382–400.

11 K. R. James, Convergence of matrix iterations subject to diagonal dominance, *SIAM J. Numer. Anal.* **10** (1973), 478–484.

12 K. R. James and W. Riha, Convergence criteria for successive overrelaxation, *SIAM J. Numer. Anal.* **12** (1974), 137–143.

13 U. Kulisch, Über reguläre Zerlegungen von Matrizen und einige Anwendungen, *Numer. Math.* **11** (1968), 444–449.

14 Jorge J. Moré, Nonlinear generalizations of matrix diagonal dominance with application to Gauss-Seidel iterations, *SIAM J. Numer. Anal.* **9** (1972), 357–378.

15 Karl Hans Müller, Zum schwachen Zeilensummenkriterium bei nichtlinearen Gleichungssystemen, *Computing* **7** (1971), 153–171.

16 A. M. Ostrowski, Über die Determinanten mit überwiegender Hauptdiagonale, *Comment. Math. Helv.* **10** (1937), 69–96.

17 Werner C. Rheinboldt, On *M*-functions and their application to nonlinear Gauss-Seidel iterations and to network flows, *J. Math. Anal. Appl.* **32** (1970), 274–307.

18 F. W. Schäfke, Zum Zeilensummenkriterium, *Numer. Math.* **12** (1968), 448–453.

19 O. Taussky, A recurring theorem on determinants, *Am. Math. Monthly* **56** (1948), 672–676.

20 Richard S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

21 Wolfgang Walter, Bemerkungen zu Iterationsverfahren bei linearen Gleichungssystemem, *Numer. Math.* **10** (1967), 80–85.

22 David M. Young, *Iterative Solution of Large Linear Systems*, Academic, New York, 1971.

23 Robert Beauwens, Semi-strict diagonal dominance, *SIAM J. Numer. Anal.* (to appear).

24 D. H. Jacobsen, Factorization of symmetric *M*-matrices, *Linear Algebra Appl.* **9** (1974), 275–278.

25 J. A. Meijerink and N. A. Van der Vorst, Iterative solution of linear systems arising from discrete approximation to partial differential equations, *J. Computational Physics* (to appear).

26 P. N. Shivakumar and Kim Ho Chew, A sufficient condition for nonvanishing of determinants, *Proc. Amer. Math. Soc.* **43** (1974), 63–66.

Received 26 July 1974