On Diagonal Dominance Arguments for Bounding $\|A^{-1}\|_\infty$

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ABSTRACT

In a recent paper by J. M. Varah, an upper bound for $\|A^{-1}\|_\infty$ was determined, under the assumption that A is strictly diagonally dominant, and this bound was then used to obtain a lower bound for the smallest singular value for A. In this note, this upper bound for $\|A^{-1}\|_\infty$ is sharpened, and extended to a wider class of matrices. This bound is then used to obtain an improved lower bound for the smallest singular value of a matrix.

1. INTRODUCTION

In a recent paper, Varah [5] established

**Theorem A.** Assume that $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is strictly diagonally dominant (cf. [6, p. 23]), and set

$$\alpha = \min_{1 \leq i \leq n} \left\{ |a_{ii}| - \sum_{j \neq i}^n |a_{ij}| \right\}.$$  

Then

$$\|A^{-1}\|_\infty \leq \frac{1}{\alpha}. \quad (1)$$

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2. MAIN RESULTS

\[(N \ni t \ni |a|) = \max \{\infty \|\lambda\| \mid a < n(V)\}_{\infty} : 0 < n \}\end{equation}\]

and denote the possibly empty set \(\Omega = \bigcap_{\gamma \in \mathcal{Y}} t^{(\gamma)}\).

\[(\gamma)\quad \text{for any } \gamma \in \mathcal{Y},\text{ denote by } Y \subseteq \mathcal{Y} = \bigcap_{\gamma \in \mathcal{Y}} t^{(\gamma)}\text{ the collection of all real column vectors } \mathbf{a} \text{ with } a < n(V)\text{ for all } N \ni t \ni |a| \leq 1.\]

We first introduce some notation. Let \(B\) be a positive integer with \(n < B\).

\[1 - \left(\|B - I\|\right) > \|B - I\| \iff \|B - I\| \leq \frac{1}{2}\]

Theorem 2. Assume that \(V \in \mathcal{C}\) and \(V \not\ni f\) are both strictly

**THEOREM 2.** Assume that \(V \in \mathcal{C}\) and \(V \not\ni f\) are both strictly
characterizations (cf. [2, 3, 7]) of a nonsingular $M$-matrix, one states that

$\mathcal{M}(A)$ is a nonsingular $M$-matrix if and only if the set $U_A$, as defined in (4), is

nonempty, so that the following statements are all equivalent:

$$
\begin{align*}
A & \text{ is a nonsingular H-matrix;} \\
\mathcal{M}(A) & \text{ is a nonsingular M-matrix;} \\
U_A & \text{ is nonempty.}
\end{align*}
$$

(5)

Thus, assuming that $A$ is a nonsingular $H$-matrix implies from (4) and (5) that

$$
f_A(u) := \min_{i \in N} \left\{ (\mathcal{M}(A)u)_i \right\} > 0 \quad \text{for any } u \in U_A.
$$

(6)

It is readily seen that $f_A$ is continuous on the set $U_A$, and that $f_A$ can be

extended continuously on $\overline{U}_A$, the closure of $U_A$. However, $f_A$ necessarily

vanishes on $\partial U_A$, the boundary of $U_A$, so that

$$
0 < \max\left\{ f_A(u) : u \in \overline{U}_A \right\} = f_A(\hat{u}) \quad \text{for some } \hat{u} \in U_A.
$$

As we shall see, $\hat{u}$ will be explicitly given in (11).

This brings us to

**Lemma 1.** If $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ is a nonsingular $H$-matrix, then

$$
\|A^{-1}\|_{\infty} \leq \frac{1}{\max\left\{ f_A(u) : u \in \overline{U}_A \right\}}.
$$

(7)

**Proof.** For any $u \in U_A$, it follows from (3) and (4) that

$$
|a_{i,i}u_i - \sum_{j \in N_i} |a_{i,j}|u_j| > 0, \quad i \in N.
$$

With $D := \text{diag} [u_1, u_2, \ldots, u_n]$, the above inequalities imply simply that

$A \cdot D = [a_{i,j} \mu_i]$ is strictly diagonally dominant. It therefore follows from

Theorem $A$ that

$$
\| (A \cdot D)^{-1} \|_{\infty} \leq \frac{1}{f_A(\hat{u})}.
$$
Note that by definition an element of \( V \)

\[
\left\{ \forall a \in \mathbb{R}^n \middle| \forall f \in \mathbb{F} \right\} \max_a \{ n \} \Rightarrow \left\{ \forall a \in \mathbb{R}^n \middle| \forall f \in \mathbb{F} \right\} \sup
\]

where

\[
\forall a \in \mathbb{R}^n \text{ for any } \max_a \left\{ \forall a \in \mathbb{R}^n \middle| \forall f \in \mathbb{F} \right\} \Rightarrow \sup_{1-a} \| 1 \| \]

(6)

Next, note that the result of Lemma 1 applies equally well to every matrix in the set \( V \). This, we see that Theorem A is a special case of Lemma 1.

Note that if \( A = \mathbb{1} \) then the desired result of (7) yields the desired result of (9).

Then, with the above-mentioned properties of \( A \), it follows that minimizing

\[
\forall a \in \mathbb{R}^n \text{ for any } \max_a \left\{ \forall a \in \mathbb{R}^n \middle| \forall f \in \mathbb{F} \right\} \Rightarrow \sup_{1-a} \| 1 \|
\]

above inequalities then gives the last relation following from the normalisation in (4). Combining the

\[
\sup_{1-a} \| 1-a \| = \left\{ \forall a \in \mathbb{R}^n \right\} \max_a \| 1-a \| = \left\{ \forall a \in \mathbb{R}^n \right\} \max_a \left\{ \forall a \in \mathbb{R}^n \right\} \left\{ \forall a \in \mathbb{R}^n \right\} \max_a \| 1-a \| \]

but

\[
\left\{ \forall a \in \mathbb{R}^n \right\} \max_a \| 1-a \| = \sup_{1-a} \| 1 \| \]

Next, write \( A \). Then, as known, \( \left\{ \forall a \right\} = \sup_{1-a} \| 1 \| \)

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It is now natural to ask if equality holds throughout (9). That this is so is proved in

**Theorem 1.** If \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is a nonsingular \( H \)-matrix, then

\[
\sup \{ \| B^{-1} \|_\infty : B \in \Omega_A \} = \| [\mathcal{M}(A)]^{-1} \|_\infty = \frac{1}{\max \{ f_A(u) : u \in U_A \}}. \tag{10}
\]

**Proof.** The hypothesis implies [cf. (5)] that \( \mathcal{M}(A) \) is a nonsingular \( M \)-matrix. Hence, with \( \xi = [1, 1, \ldots, 1]^T \), define \( \hat{u} \) by

\[
\hat{u} := \frac{[\mathcal{M}(A)]^{-1} \xi}{\| [\mathcal{M}(A)]^{-1} \xi \|_\infty}. \tag{11}
\]

Since \( \mathcal{M}(A) \) is a nonsingular \( M \)-matrix, it is known (cf. [4]) that \( [\mathcal{M}(A)]^{-1} \) has only nonnegative entries, whence \( \hat{u} > 0 \). Moreover, as \( \mathcal{M}(A) \cdot \hat{u} = \xi / \| [\mathcal{M}(A)]^{-1} \xi \|_\infty > 0 \), we know that \( \hat{u} \) is an element of \( U_A \). Hence, from the definition in (6), we deduce that

\[
f_A(\hat{u}) = \frac{1}{\| [\mathcal{M}(A)]^{-1} \xi \|_\infty} = \frac{1}{\| [\mathcal{M}(A)]^{-1} \|_\infty}.
\]

On the other hand, we know from (9) that

\[
\| [\mathcal{M}(A)]^{-1} \|_\infty \leq \sup \{ \| B^{-1} \|_\infty : B \in \Omega_A \} \leq \frac{1}{\max \{ f_A(u) : u \in U_A \}} \leq \frac{1}{f_A(\hat{u})},
\]

whence, with the previous equality, the desired result of (10) follows.

Of course, the same analysis applies directly to \( A^T \), since \( A \) is a nonsingular \( H \)-matrix if and only if \( A^T \) is. Thus, since \( \| A \|_1 = \| A^T \|_\infty \), we have as an immediate consequence of Theorem 1 the following

**Corollary 1.** If \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is a nonsingular \( H \)-matrix, then

\[
\sup \{ \| B^{-1} \|_1 : B \in \Omega_A \} = \| [\mathcal{M}(A)]^{-1} \|_1 = \frac{1}{\max \{ f_A^T(u) : u \in U_A^T \}}. \tag{12}
\]
The author is indebted to Professor Can de Boor for a clever observation
which improved this note.

Third case

Theorem 1, Lemma 1 remains an open question for the block parti-

called in [10], unless the equation of equality is considered.

Theorem 2 is also possible, but the analogous theorem of coordinate

shows of Theorems 1 and 2 similar extensions of Lemma 1, Theorem 1, and

We finally remark that Theorem 2 and its block diagonal locally domain exten-

3. REMARKS

for any \( 0 \leq \lambda \leq 1 \) and

\[
\frac{\lambda}{1 - \lambda} \left( 1 - \lambda \right) \leq \frac{\max}{1 - \lambda} \left( 1 - \lambda \right)
\]

Theorem 2, a non-singular H-matrix, then

3. REMARKS

For any \( \lambda \leq 1 \) and

\[
\frac{\lambda}{1 - \lambda} \left( 1 - \lambda \right) \leq \frac{\max}{1 - \lambda} \left( 1 - \lambda \right)
\]

which is a non-singular H-matrix. In this case

\[
\begin{bmatrix}
\frac{\lambda}{1 - \lambda} \\
\frac{\max}{1 - \lambda}
\end{bmatrix} \leq \frac{\max}{1 - \lambda} \left( 1 - \lambda \right)
\]

We remark that the second inequality of (13) cannot hold in general be

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REFERENCES

1. J. H. Ahlberg and E. N. Nilson, Convergence properties of the spline fit, J. SIAM
2. K. Fan, Topological proof for certain theorems on matrices with nonnegative
3. Miroslav Fiedler and Vlastimil Ptak, On matrices with nonpositive off-diagonal
4. A. M. Ostrowski, Über die Determinanten mit überwiegender Hauptdiagonale,
5. J. M. Varah, A lower bound for the smallest singular value of a matrix, Linear

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